Gage's New Mathematical Series

FIRST YEAR IN ALGEBRA

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PREFACE

THIS Book consists of the first section of the Authors' larger work entitled "Elements of Algebra." As its name implies, it contains material suitable in quantity and quality for a student's first year's work in Algebra. The following are some of its leading characteristics.

A BOOK FOR BEGINNERS. The opening definition connects the subject with Arithmetic. The succeeding definitions refer to the simplest numerical operations. Each definition states clearly the elementary ideas needed at the beginning, leaving, in many cases, the more difficult conceptions to be considered after some progress has been made. The illustrations are taken from objects and processes with which young students are familiar, and where possible are made evident to the eye by diagrams.

INDUCTIVE IN PLAN. Each new operation is introduced by simple numerical examples and clearly connected with arithmetical processes with which the student is already familiar. Knowledge already obtained is thus utilized in acquiring further knowledge. Rules are given as the result of a series of observations and are consequently placed at the end of an investigation — not at the beginning. The examples in each exercise are arranged on the same general plan, *i.e.*, they lead gradually from the simple to the more complex.

COMPLETE IN ITSELF. The book containing, as it does, all the elementary operations, together with their applications to the solution of equations of one and two unknown quantities, furnishes a working knowledge of the subject sufficient for many of its most important applications. The form in which it is issued, and the matter it contains, will in the Authors' opinion, be found satisfactory to the large class of students, both in city and country, who have but a short time to devote to the study of Algebra.

THE AUTHORS.

CONTENTS

CHAPTER		PAGE
I.	DEFINITIONS AND NOTATION	9
	Powers and Roots	16
	General Numbers	18
	Positive and Negative Numbers	21
II.	ADDITION AND SUBTRACTION	
	Addition	26
	Addition of Polynomials	31
	Subtraction	33
	Brackets	39
III.	MULTIPLICATION AND DIVISION	
	Multiplication	42
	Multiplication of a Polynomial by a Monomial	47
	Multiplication of Polynomials by Polynomials	50
	Division	54
	Division of Polynomials by Polynomials	57
	Literal Coefficients	61
IV.	SIMPLE EQUATIONS: ONE UNKNOWN QUANTITY	65
	Problems	70, 79
	Equations with Fractions	76
V.	SIMPLE EQUATIONS: TWO UNKNOWN QUANTITIES	84
	Problems	90
	Examination Papers	94

ELEMENTS OF ALGEBRA

CHAPTER I

DEFINITIONS AND NOTATION

1. Algebra. Algebra is a kind of advanced Arithmetic in which letters are employed as well as figures to represent numbers.

2. Numbers. Numbers as used in Algebra may be either **particular** or **general**; the former are represented by figures, the latter by letters.

Figures and letters used to represent numbers are called **numerical symbols.**

3. Particular Numbers. A particular number is any one definite number which may be specified. Such numbers are represented as in arithmetic by figures, since a figure, 3 for example, has always the same meaning or value.

4. General Numbers. A general number is one to which no definite value has been assigned. It is frequently some one number, at present unknown, whose precise value we wish to find. Such a number is represented by a letter, a, b, c, etc., to which we may assign any value whatever; but each letter is assumed to represent the same value throughout any one example or problem.

5. Signs. The signs $+$, $-$, $\times$, $\div$, $=$, used in Arithmetic to denote the operations of addition, subtraction, multiplication and division and the sign of equality, are used without change of meaning in Algebra.

These signs, excepting $=$, together with others to be explained, are called **symbols of operation.**

6. The sign of multiplication is usually omitted between two letters or between a figure and a letter, and the two symbols are placed side by side.

Thus ab means $a \times b$; $5abc$ means $5 \times a \times b \times c$, etc.

7. The sign for division, $\div$, is not very frequently used in algebra; the dividend is usually written over the divisor in the form of a fraction instead.

Thus $a \div b$ is usually written $\frac{a}{b}$, each expression indicating that the number denoted by a is to be divided by the number denoted by b.

8. Addends. Numbers represented by letters or by figures and connected by the sign $+$, denoting addition, are called **addends,** and the result of the addition is called the **sum.** Thus $a+b$ denotes that the numbers represented by a and b are to be added; a and b are consequently called addends; similarly x and 5 in the expression $x+5$, are addends.

9. Factors. Two or more numbers, represented by letters or figures which are to be multiplied, are called **factors,** and the result of the multiplication is called the **product.**

Thus ab consists of two factors; $5abc$ consists of four factors, etc.

10. Addends repeated. When a number represented by a letter is to be taken two or more times as an addend, we

write the letter once with a figure before it to show how many times it is to be so taken.

Thus $a + a = 2a$, $a + a + a = 3a$, and so on to any extent. If $a = 5$, then $2a = 10$, $3a = 15$, etc.

11. Coefficient. A figure used to show how many times another number is to be taken as an addend, is called a **coefficient** of that number.

Thus the 2 and the 3 in the preceding Art. are coefficients of the number a.

A coefficient may also be regarded as a multiplier, and if so, it and the following number are then called factors.

When a letter, considered as an addend, has no coefficient written, 1 is always to be understood.

Where a letter is used to represent a multiplier it is called a **literal coefficient.**

12. Factors repeated. When a number represented by a letter or a figure is to be taken two or more times as a factor, we write the number once with a small figure above and to the right of it, to show how many times it is to be so taken.

Thus $a \times a = a^2$, $a \times a \times a = a^3$, and so on to any extent. If $a = 5$, then $a^2 = 25$, $a^3 = 125$, etc.

13. Exponent. A number used to show how many times another is to be taken as a factor, is called an **exponent** or **index.**

When a letter, considered as a factor, has no exponent written, 1 is always to be understood.

Thus the 2 and the 3 in the preceding Art. are exponents of the number a.

14. Square and Cube. The product obtained by two equal factors is called a **square** because the area of a

square is the product of the two equal factors representing two adjacent sides.

The product of three equal factors is called a **cube** because the volume of a cube is the product of the three equal factors representing three adjacent edges.

The expressions a^2 and a^3 are read "a squared" and "a cubed."

15. Expression. A collection of algebraic symbols representing numbers is called an **expression.**

Thus $3a^2$, $5ab$, $2x - 3y$, etc., are algebraic expressions.

16. Terms and Signs. The parts of an algebraic expression connected by the signs + and − are called **terms.** Each term has a **sign** and is usually composed of factors. Where no sign is written before the first term of an expression, the sign + is understood.

Thus $4x^2 - 5xy + 6y^2$ is an algebraic expression consisting of three terms; each term consists of a coefficient or numerical factor and two literal factors.

The first and third terms have the sign +, and the second has the sign − ; if the order of the terms be changed, each term must be preceded by its own sign. The preceding expression might have been written $6y^2 + 4x^2 - 5xy$ without change of meaning.

17. Like Terms. Like terms are those which differ only in their numeral coefficients.

Thus $4ax$ and $6ax$, $3b^2y$ and $5b^2y$ are pairs of like terms, but $3ax$ and $3ay$, $5a^2b$ and $7ab^2$ are pairs of unlike terms.

18. Names. An expression consisting of but one term is called a **monomial**. Expressions consisting of two or more terms are usually called **multinomials** or **polynomials.** Sometimes, however, the words **binomial** and

trinomial are used to specify expressions of two and three terms respectively.

19. Brackets. Brackets are pairs of symbols used to combine two or more separate terms into a single term, or a single factor of a term.

Thus $a + (b - c)$ is an expression consisting of two terms of which the first term is a, and the second is $(b - c)$; $x^2 - (a + b)x + ab$ is an expression of three terms each of which has two factors, $(a + b)$ being a single factor of the second term.

A second pair of brackets may be used to enclose terms, one or more of which is enclosed by a first pair, and so on to any extent. The two parts composing a pair are of the same shape. The forms ( ), { } and [ ], are those in general use.

A line, called a Vinculum, drawn over a number of terms, serves the purpose of a pair of brackets.

Thus $a - \overline{b - c}$ means the same as $a - (b - c)$.

20. Sign with brackets. A letter or a figure written beside a bracket, or two pairs of brackets written with no sign between them, indicates multiplication. Thus $3(a + b)$ indicates that the sum of a and b is to be multiplied by 3; $(a + b)(x + y)$ means that the sum of x and y is to be multiplied by the sum of a and b. Each of the preceding expressions is a monomial consisting of two factors.

21. Order of performing operations. When several operations are to be performed, it is necessary to observe the proper order in performing them.

The multiplication of the factors comprising the several terms must precede the addition or subtraction indicated by the signs between the terms.

Thus in the expression $a+bc$ two operations are indicated, one of addition, indicated by the sign + between a and b, and one of multiplication indicated by writing the letters b and c side by side, but the multiplication must precede the addition.

Thus if $a=2$, $b=3$, $c=5$, then $a+bc=2+15=17$.

If we desired to have the addition performed before the multiplication it should be written thus $(a+b)c$, and the result would then be $(2+3)5=25$.

Again, $3a^2$ denotes two operations, squaring the a and multiplying the result by 3. These operations reversed should be written $(3a)^2$.

Thus if $a=5$, then $a^2=25$ and $3a^2=75$, whilst $(3a)^2 = (15)^2 = 225$.

22. Examples on symbols 1 and 0. Two symbols of number, 1 and 0, deserve careful attention. As a factor, 1 produces no effect and may consequently be omitted, but as an addend it must be counted. As an addend, 0 produces no effect and may be omitted, but as a factor it produces 0 as product.

Ex. 1. If $a=1$, then $5a=5$, $3ab=3b$, but $a+5=6$, $a^2=1\times 1=1$, etc.

Ex. 2. If $m=0$, then $5m=0$, $mab=0$, but $a+m=a$.

Ex. 3. If $a=1$, $b=2$, $c=3$, $m=0$,

then
$$\begin{aligned} a^3+a^2(b+c)+5(b^2-ac)-mb^2c^2 &= 1+1(5)+5(4-3)-0(4)(9)\\ &=1+5+5-0\\ &=11.\end{aligned}$$

Ex. 4.
$$\begin{aligned} b^2c^2\left(\frac{1}{a}+\frac{1}{b}-\frac{2}{c}\right) &= (4)(9)\left(\frac{1}{1}+\frac{1}{2}-\frac{2}{3}\right)\\ &=36\left(\frac{5}{6}\right)=30.\end{aligned}$$

EXERCISE I

1. If $a=4$, write down the values of
$$2a,\ a^2,\ 3a,\ a^3,\ 3a^2,\ (3a)^2.$$

2. If $a=3$, $b=5$, find the value of
$$ab,\ a^2b,\ ab^2,\ a^2b^2,\ a^2+b^2,\ (a+b)^2.$$

3. If $a=1$, $b=5$, find the value of
$$a+b,\ ab,\ 2a+b,\ 2(a+b),\ 3a^2b,\ 5(b-3a)^2.$$

4. If $a=1$, $b=2$, $c=3$, find the values of
$$a+bc,\ ab+c,\ (a+b)c,\ a(b+c),\ abc.$$

If $a=1$, $b=2$, $c=3$, $d=4$, $e=5$, $m=0$, find the value of

5. $3a+b-c+d$.
6. $2b+c-d+5$.
7. $ab+bc+ca$.
8. $2bc-cd+de$.
9. $a^2+b^2+c^2-ae$.
10. $abc+bcd+cde$.
11. 3[illegible]$^3+2e^2-b^2c^2$.
12. $b^2(c-a)+b(c-a)^2$.
13. $2(b+c)^2-(m+e)^2$.
14. $4e^2-\{(e-c)^2+bd\}$.
15. $(3b-e)(bc-a^2+m)$.
16. $(e^2-bcd)(c^2+d^2-e^2)$.
17. $\left(\frac{1}{a}-\frac{1}{c}\right)\left(\frac{1}{b}-\frac{1}{d}\right)$.
18. $bcd\left(\frac{1}{b}+\frac{2}{c}-\frac{3}{d}\right)$.
19. $\frac{1}{a}(b+c)+\frac{1}{b}(c+a)+\frac{1}{c}(a+b)$.
20. $\frac{a+b}{c}+\frac{b+c}{d}-\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{1}{a}-\frac{1}{b}-\frac{1}{a+b}\right)$.
21. $\frac{8a^2+3b^2}{a^2b^2}+\frac{4c^2+6b^2}{c^2-b^2}-\frac{c^2+d^2}{e^2}$.
22. $\frac{(a+b)(c+d)}{ab+cd}+\frac{(e-a)(e-b)}{b(e-c)}-\frac{(a+b+c)d}{c+d+e}+\frac{bcde}{abcd}$.

POWERS AND ROOTS

23. Powers. A **power** of a number is the product obtained by taking the given number two or more times as a factor.

Thus $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, etc., are the successive powers of 2, and are named the "square," "cube," "fourth power," "fifth power," etc.

24. Roots. A **root** of a number is one of two or more equal factors whose product is the given number.

25. Square Root. The **square root** of a number is one of two equal factors whose product is the given number and is indicated by the sign $\sqrt{}$ placed over the number.

Thus since $3^2 = 9$, $\sqrt{9} = 3$; $5^2 = 25$, $\sqrt{25} = 5$, etc.

26. Cube Root. The **cube root** of a number is one of three equal factors whose product is the given number and is indicated by the sign $\sqrt[3]{}$ placed over the number.

Thus since $2^3 = 8$, $\sqrt[3]{8} = 2$; $5^3 = 125$, $\sqrt[3]{125} = 5$, etc.

The fourth and higher roots are defined and indicated in a manner similar to the preceding, but they are not of frequent occurrence. The sign $\sqrt{}$ is a corruption of *r* in "radix" and is called the **Radical Sign.**

27. It will be observed that whilst the square, cube, or any power of a given number may be found by successive multiplication, but few numbers have exact roots. The various powers of the smaller numbers as indicated in the following exercise, should be written out and learned, and then the corresponding roots will be known at sight.

EXERCISE II

1. Write the squares and the cubes of all the whole numbers from 1 to 12 inclusive.

2. Of what numbers are 27, 125, 343, and 512 the cubes?

3. Write down the values of the following:

$$\sqrt{49}, \quad \sqrt{121}, \quad \sqrt[3]{216}, \quad \sqrt[3]{729}, \quad \sqrt[3]{1728}.$$

4. If $a=3$, $b=4$, $c=5$, find the values of

$$\sqrt{a^2+b^2}, \quad \sqrt{c^2-a^2}, \quad \sqrt[3]{a^3+b^3+c^3}, \quad \sqrt{(a+b)^2-c^2+1}.$$

5. Write out and learn the fourth powers of the whole numbers from 1 to 5; also the fifth and sixth powers of 1, 2 and 3.

6. Express 64 as a power of 4 and as a power of 2.

7. Express 81 and 729 as powers of 3, and of 9.

8. Express 625 and 256 as fourth powers.

9. If $x=2$, $y=3$, find the values of

$$x^5, \quad 5^x, \quad x^{y+1}, \quad (x+1)^y, \quad (y-x)^x, \quad (x+y)^{xy}.$$

10. Divide 16 into two equal addends, and into two equal factors.

11. What values of x will make

$$2x=64, \quad x^2=64, \quad 3x=27, \quad x^3=27?$$

12. How many twos make 8 if the twos are addends? if they are factors?

13. What values of x will make

$$2x=8, \quad 2^x=8, \quad 3x=81, \quad 3^x=81?$$

14. If $3x=27$, find the value of $4x$.

15. If $3^x=27$, find the value of 4^x.

16. If $x=10$, $y=12$, $a=5$, $b=1$, find the values of

$$(x+a)(\sqrt{3y}+b)+(y-b)(\sqrt[3]{a^2+2b+x})$$

and
$$\sqrt{xy+b^2}-\sqrt[3]{ax+by}+\tfrac{1}{2}(a-b).$$

17. If $2s=a+b+c$, find the value of

$\sqrt{s(s-a)(s-b)(s-c)}$, when $a=3$, $b=4$, $c=5$, and when
$$a=5,\ b=12,\ c=13.$$

18. If $2s=a+b+c+d$, find the value of

$\sqrt{(s-a)(s-b)(s-c)(s-d)}$, when $a=1\tfrac{1}{2}$, $b=3\tfrac{1}{2}$, $c=4\tfrac{1}{2}$,
$$d=5\tfrac{1}{2}.$$

GENERAL NUMBERS

28. The object of the following exercise is to familiarize the learner with the representation of numbers by letters. The brevity and simplicity of this new mode of expressing numerical relations, and its power to assist in the solution of problems, will soon be evident.

Such expressions as "the sum of any two numbers," "the product of any two numbers," etc., may be very briefly expressed by algebraic symbols, using a and b to denote numbers, as shown by the following examples:

Ex. 1. The sum of any two numbers is $a+b$.

Ex. 2. The product of any two numbers is ab.

Ex. 3. The sum of the squares of any two numbers is
$$a^2+b^2.$$

Ex. 4. The square of the sum of any two numbers is
$$(a+b)^2.$$

Ex. 5. The square of the difference of any two numbers is
$$(a-b)^2.$$

The student should verify the truth of the preceding statement by substituting for a and b any numerical values, giving the larger value to a, to make the subtraction possible.

EXERCISE III

1. Write the sum and the product of x and 5.

2. How much greater is 10 than 7? 10 than x?

3. A boy is n years old; how old will he be in 2 years? in x years?

4. A father is n years older than his son; how old is he when his son is 5 years old? when his son is y years old?

5. Tom has x marbles, Dick has as many and two more. How many have both together?

6. A rectangle is a inches long and b inches wide. How many inches around it? How much greater is its length than its width?

7. A person having \$$m$ in cash buys two articles worth \$$p$ and \$$q$ respectively. Write down the number of dollars he has left if he pays for them in succession. If he pays for them together.

8. How much will n books cost at \$3 each? at \$$x$ each?

9. A man works n days at \$2 per day and p days at \$3 per day; how many dollars has he in all?

10. How many cents in \$$x$? How many dollars in x cents?

11. A man having \$$b$ in cash buys n articles worth x cents each. How many cents has he left? How many dollars?

12. How many inches in x feet and y inches? In x yards and y feet?

13. A train runs m miles per hour. How many miles will it run in 5 hours? In x hours? In p minutes?

14. How long would it take to walk m miles at 4 miles per hour? at x miles per hour?

15. A man works q hours a day for n days and p hours a day for m days. How many hours does he work and how many dollars will he receive for it at x cents per hour?

16. A rectangle is x inches long and y inches wide. How many feet in its perimeter? How many square inches in its area and how many square feet?

17. A square is x inches on a side, and a rectangle is 3 inches longer and 2 inches narrower than the square. Find the perimeter of the rectangle in inches and in feet.

18. A boy is x years old and his brother is y years old; find the sum and the difference of their ages after 5 years, the former being the elder.

19. Write a number consisting of 7 tens and 5 units. Wr[illegible] one containing x tens and y units.

20. What value of x will make

$$2x+3=11, \quad 5x-2=53, \quad 3x^2+1=76 ?$$

21. A boy has x ten-cent pieces, as many quarters and 2 more; how many cents has he in all?

22. In the preceding example, if the coins mentioned are together worth \$4, what number does x represent?

23. The edge of a cube is x inches; find the sum of the areas of all its faces. If the sum of the areas of the faces is 54 square inches, find the value of x.

24. A block is x feet long, y feet wide and z feet thick. How many cubic feet in it? How many square feet in all

its faces? How many feet in the sum of the lengths of all its edges?

25. If n stands for any whole number, then $2n$ represents an even number. Why? Write two expressions containing n, each of which will be an odd number.

26. The divisor is x, the quotient is y, and the remainder is r; what is the dividend?

Express in algebraic symbols the following statements:

27. The square of the sum of any two numbers is equal to the sum of their squares together with twice their product.

28. The square of the difference of any two numbers is equal to the sum of their squares less twice their product.

29. The difference of the cubes of any two numbers, divided by the difference of the numbers, is equal to the sum of the squares of the same number together with their product.

30. The product of the sum and the difference of any two numbers is equal to the difference of their squares.

31. Verify the truth of each of the four preceding statements by substituting numerical values for each of the two letters in each.

POSITIVE AND NEGATIVE NUMBERS

29. So far, the numbers with which we have been concerned are the ordinary numbers used in Arithmetic, or letters used to represent them. These numbers are 1, 2, 3, 4, 5, etc., counting from zero upward. These are called positive numbers and may be written with the positive sign attached to them. In order to perform the

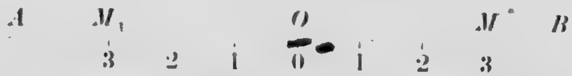
operations of algebra, however, we make use of another set of numbers, found by counting from zero in the **opposite direction.** These numbers are called negative numbers, and have the sign, $-$, prefixed to them. The two sets of numbers when arranged in a series each differing from the adjacent ones by unity, appear as follow :

$$\ldots -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ +4,\ +5 \ldots$$

30. While these negative numbers have no significance in ordinary arithmetic, a few concrete examples will show how they may be interpreted in algebra.

Ex. 1. If A possesses \$10 and by investing it makes a gain of \$5, thus increasing his \$10 to \$15, the \$5 gain may be considered a positive quantity, as it **increases** the amount possessed by A. If, on the other hand, he loses \$5 by the investment, thus diminishing his \$10 to \$5, the \$5 loss may be considered a negative quantity, as it **decreases** the amount possessed by A. The \$5 loss acts upon the original \$10 in the **opposite direction** to the \$5 gain. If, therefore, gain be considered positive, loss may be considered negative.

Ex. 2. If we desire to measure a distance of 3 inches along the line AB, from a fixed point O, the zero point for measurement,



we may measure in the direction OB or the **opposite direction** OA. To distinguish between the two, we may consider the direction OB positive, and the direction OA negative. OM is thus equal to $+3$ inches and OM_1 to -3 inches.

31. From the examples of the preceding Art. it will be observed that in the complete representation of concrete quantities by numbers, three elements must be clearly specified: the unit, the number of units, and the mode or direction of measurement.

Thus 5 dollars gain, 3 inches to the right, are examples of concrete quantities accurately specified. The units are "dollars" and inches; the numbers of units are 3 and 5; and the directions of measurement are indicated by the phrases, "gain" and "to the right."

Conversely if a dollar gain be the unit, $+5$ means 5 dollars gain, -3 means 3 dollars loss; if a mile to the north is the unit, $+7$ means 7 miles north, -4 means 4 miles south. Thus concrete quantities may always be represented by numbers, and numbers (taken with a unit of measurement) represent concrete quantities.

32. When performing the operations of algebra we may think of the numbers only, without reference to the concrete units, dollar, inch, and we have the series of numbers given above, viz.:

$$\ldots -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ +1,\ +2,\ +3,\ +4,\ +5 \ldots$$

In this series observe

1. We can start at any point and count without limit in either direction, whilst with purely arithmetical numbers we must stop at 0.

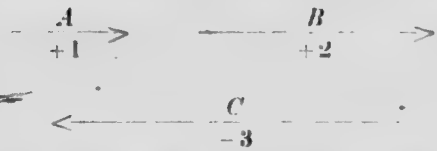
2. The direction in which the positive numbers increase is the positive direction, and the opposite is the negative direction. From -5 to -4, -3, etc., the direction is positive; similarly from $+5$ to $+4$, $+3$, etc., the direction is negative.

3. When magnitude alone is considered, a negative

number and the quantity it represents are exactly equal to the corresponding positive number and quantity it represents: -3 is the same number of units as $+3$, and three dollars loss is the same amount of money as 3 dollars gain.

4. When direction alone is considered, it is customary to say that the numbers increase when proceeding in the positive direction from whatever point in the series we begin. This is equivalent to assuming that algebraically -5 is less than -4 and that any negative number whatsoever is less than zero. "Less than nothing" with reference to magnitude is absurd; with reference to position in the series of numbers the meaning is clear.

Ex. If the line A, one inch long, drawn to the right, be the unit, then the line B, 2 inches long, drawn in the same direction, is denoted by $+2$, and the line C, 3 inches long, drawn in the *opposite* direction, is denoted by -3. Had



the unit been one inch, drawn to the left, then the lines B and C would have been represented by -2 and $+3$, respectively.

EXERCISE IV

1. If a line two inches long, drawn from left to right, be the unit, what numbers will represent 3 ft. to the right? 5 yd. to the left?

2. If the unit be 3 ft. to the north, what will be represented by $+10$, -5, $-2\frac{1}{2}$, $+\frac{1}{3}$?

3. What is the unit of measurement when a tree 50 ft. high is represented by $+10$? by -5?

4. If a dollar gain be the unit, what will be represented by $+2\frac{1}{2}$, $-3\frac{1}{4}$?

5. In the preceding example what will represent a loss of \$2.75? a gain of \$3.40? \$4.12½ cash in hand? a debt of \$2.50?

6. If one day forward be the unit of time, what number would refer to *yesterday?* The day after to-morrow?

7. If the letter a represents a line of any given length drawn to the right, what would represent a line twice as long drawn to the left? three times as long drawn to the right?

8. When a dollar gain is the unit, what will represent the sum of \$3 debt and \$7 debt? \$5 cash and \$2 debt? \$7 loss and \$3 gain?

9. What must be added to \$5 debt to produce \$7 cash? to \$3 cash to produce \$2 debt?

Represent each of these sums of money by the appropriate number, a dollar cash being the unit.

10. Berlin is 13½ degrees east longitude and St. Petersburg 30½ degrees east. What will represent in degrees a journey from Berlin to St. Petersburg? St. Petersburg to London?

11. A toy balloon can just sustain a weight of six ounces. If attached to a paper basket weighing 4 oz., what would the two combined *weigh?* What weight must be attached to the balloon to make the two weigh 8 oz.? 4 oz.?

12. What change takes place in the number representing a concrete quantity when the unit is doubled? halved? direction changed?

CHAPTER II

ADDITION AND SUBTRACTION

ADDITION

33. Meaning of addition. The operations of addition and subtraction are employed in a more extended sense in Algebra than in Arithmetic, inasmuch as algebra employs two sets of numbers, the positive and the negative, whilst arithmetic employs but one. The fundamental ideas, however, in the two cases are the same; that of addition being the operation of collecting into one number that which formerly existed as two or more separate numbers, whilst subtraction is simply the process of addition reversed. This will be evident from a few simple concrete examples.

Ex. 1. A gain of \$5 in one business transaction, followed by a loss of \$2 in a second transaction, gives a net result of \$3 gain on the whole. The process of combining these two separate items into one is addition, and may be expressed thus:

$$\$5 \text{ gain} + \$2 \text{ loss} = \$3 \text{ gain}.$$

Ex. 2. A journey of five miles north, followed by a journey of 7 miles south, leaves the traveller 2 miles south from the starting point. The addition of the two journeys may be thus expressed:

$$5 \text{ mi. N.} + 7 \text{ mi. S.} = 2 \text{ mi. S.},$$

i.e., we have found a single journey equivalent in result to two specified journeys.

34. Representation of addition. If now we replace the words "gain" and "loss," "miles north" and "miles south" by the signs + and −, these examples of addition may be conveniently expressed thus:

$$(+5)+(-2)=+3$$
$$(+5)+(-7)=-2.$$

The signs + and − when used as above to denote positive and negative quantities are enclosed in brackets, with the numbers to which they refer, to distinguish them from their use to denote the operations of addition and subtraction.

When the quantities to be added are both positive or both negative, the process is the same as in arithmetic, thus:

$5 gain + $3 gain = $8 gain.

$5 loss + $3 loss = $8 loss,

which, expressed in algebraic symbols, becomes

$$(+5)+(+3)=+8$$
$$(-5)+(-3)=-8.$$

35. Mode of addition. The truth of the preceding additions is readily perceived when the concrete units "gain," "loss," etc., are expressed. When only the signs + and − are given we proceed thus:

To add $+5$ and -2, we find $+5$ on the scale of numbers, Art. 32, and from it count 2 units in the *negative* direction, as indicated by -2; the result is $+3$.

To add $+5$ and -7, begin with $+5$ and count 7 units in the negative direction, the end of the counting being -2, which is the sum required.

Similarly any algebraic numbers may be added.

From these special examples the definition of the following Art. will be readily understood.

36. Definition. The **sum** of two or more numbers is the number obtained by counting in succession the number of units in the several addends, each in the direction indicated by its sign, beginning the count from zero. The process of finding the sum is **addition.**

37. Addends may be taken in any order. If in the examples of Art. 34 we reverse the order of the addends, we get

$$(-2)+(+5)=+3$$
$$(-7)+(+5)=-2$$

as before. For beginning with -2, and counting 5 units in the positive direction, the end of the count is $+3$; similarly the same result is obtained with any pair of numbers with whichever one we begin. If, then, we have several addends, it will be easiest to combine all the positive numbers and all the negative numbers separately, which is merely arithmetical addition, and then combine the two results.

38. Rule for addition. *To add two or more numbers having* **like signs,** *add the numbers as in arithmetic, and prefix the common sign.*

To add two numbers having **unlike signs,** *take their difference as in arithmetic, and prefix the sign of the greater.*

39. Positive and negative numbers may be used as coefficients of literal expressions, with meanings derived from the definition of Art. 29.

Thus if a denotes any number, $2a$ means twice as large a number, counted in the same direction, on the scale of numbers, as a; but $-3a$ means three times as large a number, counted in the opposite direction.

The sum of any positive number and the same number taken negatively, is evidently zero.

40. If in the examples of Art. 33 we represent a "dollar gain" or "a mile to the north" by the letter a, we get by the reasoning previously given

$$(+5a)+(-2a)=+3a, \quad (+5a)+(-7a)=-2a,$$
$$(+5a)+(+3a)=+8a, \quad (-5a)+(-3a)=-8a,$$

from which we derive the rule for adding like terms.

41. Addition of terms. To add **like terms** we take the algebraical sum of the coefficients and affix the common literal factors. These literal factors may be single letters, as above, or expressions in brackets.

Thus $\qquad 5(a+b)+3(a+b)=8(a+b)$, etc.

Unlike terms can be added only by writing them with the sign of addition between them.

Thus the sum of $2a$ and $3b$ is $2a+3b$. The sum of $2x$ and $-3y$ is $2x+(-3y)=2x-3y$. These sums cannot be expressed as single terms.

42. Omission of brackets. In practice, the brackets used to distinguish the use of + and − as signs of positive and negative number, from their use as signs of addition and subtraction, are usually omitted, and the four examples of algebraical addition in Art. 40 become

1. $5-2=3$. $\qquad$ **2.** $5-7=-2$.

3. $5+3=8$. $\qquad$ **4.** $-5-3=-8$.

When written thus the first example becomes the same as an example in subtraction, which shows that to subtract 2 from 5 is the same as to add -2. The second example, though written like an example in subtraction, cannot be so considered; for 7 is greater than 5 and cannot be taken from it. This shows that the addition of a negative number, according to the definition given, is possible even when the corresponding arithmetical subtraction is impossible.

43. The addition of Algebra appears in some cases to contradict well-known facts of Arithmetic, but it is only in appearance. In Ex. 2, Art. 33, a journey of 5 miles followed by a journey of 7 miles is said to be equivalent to a single journey of 2 miles. In Arithmetic we should say the two journeys are together equivalent to one of 12 miles. Both statements are equally true. The *algebraical addition gives the position of a traveller at the end of his journey; the arithmetical addition gives the distance he has travelled.*

44. The learner should become familiar with addends written in either of the following forms:

Ex. 1. Simplify $(-3a)+(+2a)+(-5a)+(4a)$.

Combine the positive and negative coefficients separately and then combine the results.

Thus $\quad 2+4=6, \quad -3-5=-8, \quad$ then $6-8=-2$.

The given expression thus becomes $-2a$.

Ex. 2. $3a+5b-2b+4a-7b-a$

$$=(3+4-1)a+(5-2-7)b=6a-4b.$$

EXERCISE V

Add the following:

1.	$+3$	-3	$+3$	-3	$+7$	-7
	$+5$	-5	-5	$+5$	-10	$+10$
2.	$+14$	-14	$+14$	-14	$+14$	-14
	$+16$	-16	-16	$+16$	$+16$	-16
	$+20$	-20	$+20$	-20	-20	$+20$
3.	$3a$	$5b$	$7a^2$	$-12ab$	$11xy$	$3m$
	$5a$	$-6b$	$-3a^2$	$4ab$	$-2xy$	$-8m$
	$-7a$	$10b$	$-8a^2$	$-2ab$	$15xy$	$12m$

4. $4x^2+(-3x^2)+(-7x^2)+(+11x^2)+(-x^2)$.

5. $(-3x^2)+(-5x^2)+(-3y^2)+(+7y^2)+(+x^2)+(-y^2)$.

6. $3m^2-5n^2-6n^2-8m^2+n^2+m^2-2m^2$.

7. $-2ab-5ab+3ab+7ab+ab-6ab-ab$.

8. $5a^2b+3ab^2-6ab^2-7a^2b+a^2b-ab^2$.

9. $3(a+b)+5(a+b)-7(a+b)-6(a+b)-(a+b)$.

10. If $a=2,\ b=-3,\ c=7$, find the value of

$$a+b+c, \quad a+b-c, \quad b-c-a.$$

11. A traveller takes three successive journeys, 10 miles east, 25 miles west and 5 miles east. Add his journeys both algebraically and arithmetically and give the meaning of the result in each case.

ADDITION OF POLYNOMIALS

45. When two or more polynomials are to be added, it is convenient to arrange the terms in columns, so that like terms shall stand in the same column. When a term is moved to a different position its sign must be taken with it. The sign of a first term may be omitted when positive, but if another term be placed before it, the sign must be restored. The columns should be added in succession, beginning at the left.

Ex. 1.

$$\begin{array}{rrr} 2a & +3b & -4c \\ 4a & -5b & +2c \\ -3a & +b & -c \\ 5a & +2b & +3c \\ \hline 8a & +b & \end{array}$$

Ex. 2.

$$\begin{array}{rrrr} 4x^2 & -xy & +2y^2 & \\ 3x^2 & +5xy & -7y^2 & \\ -x^2 & -xy & +3y^2 & \\ -6x^2 & & +y^2 & -z^2 \\ \hline & 3xy & -y^2 & -z^2 \end{array}$$

Ex. 3. Find the sum of $3(a+2b-c)$ and $4(2a-b+3c)$.

From Art. 11, $\quad 3(a+2b-c)=3a+6b-3c$

$$4(2a-b+3c)=\underline{8a-4b+12c}$$

$$\text{Their sum}=11a+2b+9c.$$

EXERCISE VI

Add:

1. $3a + b - c,\ 4a - 2b + 3c,\ -a + 5b - 6c.$
2. $5a - b + 2c,\ 3b + 4c - 2a,\ 5c - a + 2b.$
3. $6a + 2b - 5c,\ 4a + 5c - 3b,\ 2c - a - 4b.$
4. $7a - 4b - 3c,\ a + b + x,\ b - c - 5x.$
5. $4ab - ac + bc,\ bc - 4ab - ac,\ 4ac + 5ab.$
6. $x^2 - ax - 2bx,\ 2cx - 2x^2 + ax - 2bx,\ bx - cx - ax + x^2.$
7. $a + b - 2c,\ b + c - 2d,\ c + d - 2a,\ d + a - 2b.$
8. $3(a + b),\ 5(a + b),\ -2(a + b),\ a + b.$
9. $a + b - c,\ 2(a + b - c),\ -(a + b - c),\ -(a + b - c).$
10. $5(a^2 + b) + 2c,\ 3(a^2 + b) - 7c + d,\ 2c - 4d,$
    $3d - 6(a^2 + b),\ 3c + 7(a^2 + b).$
11. $4a(b + c) - 5d,\ a(b + c) + 7d,\ -3a(b + c),\ 3d,$
    $2d - 5e,\ 6e - 2a(b + c),\ 6a(b + c) - 7d + x.$
12. $7a - 3b + 5c - 10d,\ 2b - 3c + d - 4e,\ 5c - 6a - 4e + 2d,$
    $-3b - 8c + 7a - e,\ 21e - 16c + a - 5d.$
13. $3(a^2 + b^2) + 2ab,\ a^2 - 5ab + b^2,\ 10ab - 5(a^2 + b^2),$
    $3a^2 + 6ab + 3b^2,\ a^2 + b^2.$
14. $a^3 - 3a^2b + 2ab^2,\ b^3 - 3ab^2 + 5a^2b,\ 2a^2b + 5ab^2,$
    $a^3 + b^3 + 2a^2b - 5ab^2,\ 7a^2b - a^3 - 2b^3 - 5a^2b,$
    $3a^2b - 2a^3 - qb^2 + a^2b.$
15. $a^4 + a^3b - 2a^2b^2,\ 3a^3b - 5a^2b^2 - 6b^4,\ ab^3 - 3a^2b^2,$
    $2a^2b^2 - 5a^3b + a^4 + b^4,\ 4ab^3 - 2b^4 + 3a^2b^2.$
16. $3(x + y + z),\ 4(x - y + z),\ 5(x - y - z),\ 3x - y + z.$
17. $4(2x - y + 3z),\ 5(y - 2z - x),\ 7(3z - x - 2y).$
18. If $x = a + 2b + 3c,\ y = b + 2c - 3a,\ z = c - 2a + 3b,$
    find the value of $x + y + z.$
19. In the preceding example, find the value of

$$x + 2y + 3z.$$

SUBTRACTION

46. The meaning of Subtraction in Algebra follows directly from the meaning assigned to addition. In addition two addends are given and their sum is to be found. In subtraction the sum and one addend are given and the remaining addend is to be found. Thus to subtract 7 from 10 means that we are to find the number 3, which must be added to 7 to make 10. In arithmetic we cannot subtract 10 from 7, because there is no arithmetical number which being added to 10 will make 7. The negative number -3 when added to 10 makes 7; we therefore say that 10 from 7 leaves -3.

47. Definitions. When the sum of two numbers and one of them are given, subtraction is the process by which the other is found. The sum of the two numbers is called the **Minuend,** the given number is called the **Subtrahend** and the number to be found is called the **Difference.**

The difference is, therefore, the number which must be added to the subtrahend to make it equal to the minuend.

48. Let it be required to perform the following subtractions:

From	7	-3	-1	2	0
Take	-2	-5	4	5	-5
Result	9	2	-5	-3	5

To find what must be added to the subtrahend to make the minuend in each case we reason as follows:

7 is the sum of two addends, one of which, -2, is to be removed; -2 will be cancelled, or removed, by adding 2, making 9, the other addend required. The sum of -2 and 9 is 7, which proves the work correct.

Similarly -5 is one of two addends which together make -3; to -3 add $+5$, thus cancelling the given addend, and we obtain $+2$, the remaining addend or difference required.

49. Rule for subtraction. *To subtract one number from another, change the sign of the subtrahend and add it to the minuend.*

50. The truth of the several subtractions of Art. 48 will be evident to the eye by reference to the scale of positive and negative numbers, Art. 32.

Thus from

-2	to	$+7$	is	9	units in the	positive	direction	$=9$
-5	"	-3	"	2	"	"	"	$=2$
4	"	-1	"	5	"	negative	"	$=-5$
5	"	2	"	3	"	"	"	$=-3$
-5	"	0	"	5	"	positive	"	$=5$

which are the results formerly obtained.

51. The truth of the rule given for subtraction is also readily perceived by observing that the distance between any two numbers on the scale is not changed by adding the same to each of the given numbers. If, then, we add to each the subtrahend with its sign changed, the new subtrahend is 0, and the new minuend is therefore the result required. The operation here described is precisely the rule given.

52. The subtraction of like terms follows immediately from the subtraction of positive and negative numbers. Thus

From	$5x$	$3a^2b$	$-2(a+b)$	$3a$
Take	$2x$	$-2a^2b$	$-4(a+b)$	$-5b$
Result	$3x$	$5a^2b$	$2(a+b)$	$3a+5b$

53. Subtraction of terms. **Like terms** *are subtracted by the algebraical subtraction of their coefficients and annexing their common literal factors.*

Unlike terms *can be subtracted only by connecting the terms by the proper signs.*

54. The double use of the sign $-$, to denote both a negative number and the operation of subtraction, is somewhat confusing to a beginner. This is especially the case when a change is made from one meaning to the other in the same example. The two meanings, however, lead always to the same result, and are in fact only two ways of thinking and speaking of the same facts. In this connection it is important to observe the truth of the three following statements:

1. The subtraction of a number of positive units is equivalent to the addition of the same number of negative units.

2. The subtraction of a number of negative units is equivalent to the addition of the same number of positive units.

3. The negative of a negative is positive. Expressed in symbols, these statements become:

1. $a-(+b)=a+(-b)=a-b.$
2. $a-(-b)=a+(+b)=a+b.$
3. $-(-b)=+b.$

55. Illustrations of the preceding, from actual experience, are numerous:

1. A decrease in a man's income produces the same effect as an equivalent increase in his expenses.

2. A decrease in expenses is equivalent to a corresponding increase in income

3. The negative of income is expense, the negative o expense is income, *i.e.*, the negative of a negative i positive.

The student should carefully study the theory of alge braic numbers and the illustrations as here given, bu when working examples *he should think of nothing but th Rule.*

EXERCISE VII

Subtract

1.	5	-3	-5	-3	10	7
	-3	5	-3	-5	7	1
2	-8	12	0	-7	-11	$-$
	1	-2	-3	0	-7	-1
3.	$4x$	0	$14a^2$	$-2ab$	$3a$	4
	$-6x$	$-y$	$-5a^2$	$7ab$	$5b$	-3

4. $7ax-(+3ax)$, $2by-(-5by)$, $-a-(-5a)$.

5. $5x^2-7x^2$, $-2y^2-8y^2$, $-3xy+5xy$.

6. $3a^2-11a^2+7a^2-9a^2$, $5x^3+(-2x^3)-(+4x^3)-(-x^3)$

7. $4a-(-2b)+8b-(+7a)-6a-5b$.

8. If $a=3$, $b=5$, $c=7$, find the value of

$$a-b+c,\quad a-(b+c),\quad -(a+b+c).$$

9. If $a=2$, $b=4$, $c=-7$, find the value of

$$a-b,\quad b-c,\quad c-a,\quad a-(b-c).$$

10. Simplify

$$4(x-y)^2-5(x-y)^2+3(x-y)^2-11(x-y)^2-(x-y)^2.$$

11. Toronto is 44 degrees north latitude and Ri Janeiro is 23 degrees south latitude. Find by algebra ical subtraction the number of degrees Toronto is north o Rio Janeiro.

56. Subtraction of polynomials. When the subtrahend contains two or more terms, the subtraction is performed by subtracting each term in succession, *i.e.*, by changing the sign of each term of the subtrahend and adding it to the minuend. Like terms should be placed under each other as in addition.

Ex. 1.

From	$4a^3 - 3a^2b + 7ab^2 - b^3$
Take	$a^3 - 5a^2b + 8ab^2 + 2b^3$
Result	$3a^3 + 2a^2b - ab^2 - 3b^3$

The signs of the terms in the lower line should be changed in thought only and added to those above. Thus we say -1 and 4 make 3; $+5$ and -3 make 2; -8 and 7 make -1; -2 and -1 make -3, thus giving the required coefficients to which the literal factors are to be affixed.

Ex. 2. What must be added to $a^2 - b^2 - c^2 + 2ab - ac$, to make $a^2 + ab + bc + ac$?

The first expression is evidently the subtrahend; write it below the other, like terms under each other so far as possible.

From	$a^2 \qquad\qquad\qquad + ab + bc + ca$
Take	$a^2 - b^2 - c^2 + 2ab \qquad - ac$
Result	$b^2 + c^2 - ab + bc + 2ac$

EXERCISE VIII

1. From $4a - 3b + c$ take $2a + b - 3c$.
2. From $3a + 2b - 5c - 6d$ take $4a - 2b - 5c - 7d$.
3. From $-a - b + 2c$ take $b - c - a + x$.
4. From $3x^3 - 5x^2y + y^3$ take $2x^2y - 3xy^2 - x^3 + y^3$.
5. From $1 - 2x + 3x^2 - 5x^3$ take $x^3 + 3x^2 - 5x - 1$.
6. From $a^2 - b^2 - c^2 + 2bc$ take $b^2 - c^2 - a^2 + 2ac$.

7. From $4x^2 - 3xy + 7y^2 - 5xz + 6yz - z^2$
take $x^2 - y^2 - z^2 + xy - 7xz + 7yz$.

8. From $x^2 + y^2 - z^2 + 2xy - 3xz + yz$
take the sum of $yz - x^2$, $xy - z^2$, and $y^2 - 3xz$.

9. From the sum of $2x^3 - 3x^2y + y^3$ and $2xy^2 - x^3 - 4y^3$
take the sum of $x^2y + 2xy^2$ and $x^3 + y^3 - 3x^2y$.

10. What must be added to $a^2 + b^2 - c^2 + 2ab - ac - bc$
to make $ab + bc + ca$?

11. What must be subtracted from $1 - a + b + a^2b + ab^2$
to leave $a^2b - b + c + 1$?

12. What must be subtracted from the sum of
$4x^3 + 3x^2y - y^3$, $4x^2y - 3x^3$ and $7x^2y + 9y^3 - 2x^2y$
to leave $2x^3 - 3x^2y + y^3$?

13. From $5(a - b) + 6(x - y)$ take $2(a - b) - 7(x - y)$.

14. From $3(a + b) - 4(c + d) - 5(x - y) + p$
take $a + b - 5(c + d) - 7(x - y) - q$.

15. From $3(a + b - c) + 5(a - b + c) + 3(a - b - c)$
take $2a - 3b + 2(b + c - a)$.

16. From $a - b + 2c - 3d$ take the sum of

$2a + 3b - c + 4e$ $\qquad$ $b + 3c - 4d + 5e$

$2c + d - e - 5a$ $\qquad$ $d - 2e + 3a - b$.

Verify the result by subtracting in succession each expression separately.

17. From $9a - 4b - 17c - 12d + 12e$ take the sum of

$7a - 3b + 5c - 10d$ $\qquad$ $2b - 3c + d - 4e$

$5c - 6a - 4e + 2d$ $\qquad$ $-3b - 8c + 7a - e$

and verify result as in preceding example.

18. If $x = 2a + b - c$, $y = 3b + c - a$, $z = c + a - b$,
find the value of $2x - 3y - 4z$.

BRACKETS

57. Removal of brackets. The sign + preceding a bracket, indicates that the terms contained are to be added to what precedes. Now terms are added by connecting them in succession, each preceded by its own sign. Therefore

A bracket preceded by the sign + may be removed, each term retaining its sign unchanged.

58. The sign − preceding a bracket, indicates that the terms contained are to be subtracted from what precedes. Now subtraction requires that the sign of each term subtracted be changed and the result added. Therefore

A bracket preceded by the sign − may be removed, providing the sign of every term within be changed.

59. The truth of each of the following equalities should be carefully considered and verified by assuming a suitable value for each letter.

1. $a+(b+c)=a+b+c.$
2. $a+(b-c)=a+b-c.$
3. $a-(b+c)=a-b-c.$
4. $a-(b-c)=a-b+c.$
5. $a+(-b+c)=a-b+c.$
6. $a-(-b+c)=a+b-c.$

Observe that each term within a bracket has its own sign, but the term + is omitted before the first term inside a bracket. The sign preceding a bracket belongs to the expression as a whole, and is no longer needed when the bracket is removed.

60. Two or more pairs of brackets are frequently used in the same expression, one pair enclosing a portion of the terms enclosed by another pair. In such cases the two parts forming one pair must be carefully observed. It is the simplest to remove them one pair at a time, taking always the innermost. A little experience, however,

will enable the student to take them in any order, and to remove several pairs at one operation. At each step like terms should be combined to save labor in writing.

Ex. 1. $a - \{b - (c - a) + (b - c) - a\}$
$$= a - \{b - c + a + b - c - a\}$$
$$= a - 2b + 2c.$$

Ex. 2. $x - [x - a - (2a - 2x) + \{a - (a - x)\}]$
$$= x - x + a + (2a - 2x) - \{a - (a - x)\}$$
$$= a + 2a - 2x - a + (a - x)$$
$$= 2a - 2x + a - x$$
$$= 3a - 3x.$$

In Ex. 2 the outer brackets were removed each time, and in both examples like terms were combined after the removal of each pair.

61. The rules for the insertion of brackets follow immediately from the rules for their removal.

Thus $a - b + c - d - e + f = (a - b) + (c - d) - (e - f)$
$$= a - (b - c) - (d + e) + f$$
$$= a + (-b + c) + (-d - e + f)$$
$$= \{a - (b - c) - (d + e) + f\}, \text{ etc.}$$

It will be observed that a term placed in a bracket preceded by the sign + retains its own sign, but when the sign − precedes the bracket, the sign of each term is changed. The identity of the above expressions should be verified by removing the brackets from each of them.

EXERCISE IX

Remove the brackets from the following expressions and combine like terms:

1. $(a + b) + (a - b)$.
2. $(a + b) - (a - b)$.
3. $(a + b - c) - (b + c - a)$.
4. $a - (b - c) + (c - a)$.

5. $2a - b - c - (b + c - a) + (-c + 2b + 3a)$.

6. $3a - (b - 2c) - \{c - (a - b) + (a - c)\} - (2b - 2c)$.

7. $(2a - 3b) - (2 - b + c) + \{a - (b + c - a) + (c - 2a) - b\}$.

8. $x - \{a - 1 - (x - a + 1) + 2 - (a + 3)\} - (-3a + x)$.

9. $2x + [-x + a - (2a + 2x) + \{a - (a - x) + x - 2a\} - a]$.

10. $-\{(x - 2y) - (x + 3)\} - \{2 - (x - 3y) + 2x\}$
$- \{3 - (x - y)\}$.

11. $a - [2b + \{a - 2b - (a - b + c) + b\} - a - (b - c)]$.

12. $x - [y - \{z - (x - y) + z\} - (x - y + z)]$.

13. $\{(3a - 2b) + (2c - a)\} - \{a - (b - 2a) - c\}$
$+ \{a - (b + c)\}$.

14. $a - [b - (a - b) - \{a - (b - a) - b\}$
$- \{a - (b - \overline{a - b} - a)\}]$.

15. $a - [b - \{a - (b - \overline{a - b} - a) - b\} - a]$.

16. Arrange the terms of $a - b + c + d - e - f$ in alphabetical order in brackets; two terms in each pair of brackets; three terms in each pair.

17. In the same expression, place b, c, d in one pair and e, f in another pair, with the sign $-$ in front of each pair; the sign $+$ in front of each pair.

18. In the same expression enclose b and c, d and e, in small brackets, and then enclose these groups with f in an outer pair.

19. Verify the work of the preceding example by first inserting the outer pair of brackets and then inserting the two inner pairs.

20. Add $a - \{b - (c + d) + e\}$, $a - \{b + (c - d) - e\}$,
$a - [b - \{c - (d - e)\}]$ and $-\{(a - b) - (c - d)\} - e$,
and from their sum take $a - \{b - (c + a) + b\} + c$.

CHAPTER III

MULTIPLICATION AND DIVISION

MULTIPLICATION

62. In Arithmetic, when one number is multiplied by another, the former is called the **Multiplicand** and the latter the **Multiplier.** The result of the multiplication is called the **Product.**

The same terms are used in Algebra.

63. In Arithmetic the **process** of multiplication is defined as follows:

One number is multiplied by another when the former is used as an addend as many times as the number indicated by the latter.

For example, $5 \times 3 = 5 + 5 + 5 = 15.$ (1)

We have simply to extend this definition to include negative numbers, to define the process of multiplication in Algebra.

64. The use of negative numbers gives rise to three new cases of multiplication, each of which must be clearly understood:

I. A **negative** multiplicand with a **positive** multiplier.

II. A **positive** multiplicand with a **negative** multiplier.

III. A **negative** multiplicand with a **negative** multiplier.

I. The first of these is easily understood.

For example, $(-5) \times 3 = (-5) + (-5) + (-5) = -15.$

Here the sign, $-$, presents no difficulty; it merely shows the direction in which the 5 is counted. Counting from zero to -5 on the scale of numbers and repeating the counting in the same direction three times brings you to -15.

To take a concrete example: If a man travels 5 miles in the direction selected as negative, then continues 5 miles further, and again continues 5 miles further, he will finally be 15 miles in a negative direction from his starting point.

The distinction, therefore, between 5×3 and $(-5) \times 3$ is simply that the product of 5 and 3 in the one case is counted in the positive direction, and in the other case in the negative direction.

Hence $$(-5) \times 3 = -(5 \times 3) = -15. \qquad (2)$$

II. In performing a multiplication by a negative multiplier we have only to keep in mind the fact stated in the definition of a negative number, namely, that the presence of the minus sign changes the direction of counting, and the meaning of the process is quite clear.

For example, $5 \times (-3)$ means that 5 is to be multiplied by 3 and the sign of the product changed, that is, the product, 15, is to be counted in the direction opposite to that in which the multiplicand, 5, is counted.

Hence $$5 \times (-3) = -(5 \times 3) = -15. \qquad (3)$$

The same result is at once apparent if we assume the arithmetical law, that the multiplier and multiplicand can be interchanged without changing the product.

Thus $$5 \times (-3) = (-3) \times 5,$$

then from (2) $$= -(3 \times 5)$$
$$= -15.$$

Hence the rule: *To multiply a positive number by a negative number perform the ordinary arithmetical multiplication without regard to sign and give the negative sign to the product.*

III. The same principle applies when a negative multiplicand is to be multiplied by a negative multiplier.

For example, $(-5) \times (-3)$ means that the multiplicand (-5) is to be multiplied by 3, giving -15, see (2), and then the direction of counting is to be changed, making the product 15.

This may be stated thus,
$$\begin{aligned}(-5) \times (-3) &= -(-5) \times 3\\ &= -(-15)\\ &= +15. \qquad (4)\end{aligned}$$

65. The signs of the products in the four examples of the preceding Art. do not in any way depend upon the numerical value of the particular multiplicands or multipliers used, but upon their signs alone. We have, therefore, for any numbers whatsoever,

1. $(+a) \times (+b) = +ab.$ **2.** $(-a) \times (+b) = -ab.$

3. $(+a) \times (-b) = -ab.$ **4.** $(-a) \times (-b) = +ab.$

That is, the sign of the product of two numbers is + when both numbers have the same sign, and − when they have different signs. More briefly expressed, this becomes what is known as the

Rule of Signs. *Like signs give* +, *unlike signs give* −.

The sign of the product of three or more numbers may be obtained by a repeated use of this rule. Thus:

1. The product of any number of **positive** factors is **positive.**

2. The product of any *even* number of **negative factors** is **positive.**

3 The product of any *odd* number of **negative** factors is **negative.**

4. If the sign of one factor be changed the sign of the product is changed.

66. The student should observe that while the multiplicand may be either a concrete quantity or a simple number, *the multiplier must always be abstract.* It is simply a number used to count addends in one of two directions of measurement. 9.

67. Factors may be taken in any order without change of product.

Thus $2 \times 3 \times 5 = 2 \times 5 \times 3 = 5 \times 3 \times 2 =$, etc., $= 30$.

Similarly $abc = bca = cab =$, etc.

This principle enables us to combine the numerical factors from two different expressions whose product is to be found.

Thus $3a \times 4b = 3 \times a \times 4 \times b = 3 \times 4 \times a \times b = 12ab$.

When a figure and one or more letters are factors it is customary to place the figure first, and the letters in the order of the alphabet, as in the preceding example.

68. The product of two powers of the same number is formed as follows:

Since $a^2 = aa$ and $a^3 = aaa$,

we have $a^2 \times a^3 = aa \times aaa = aaaaa = a^5$.

Similarly $a^m \times a^n = a^{m+n}$, $a^m \times a^n \times a^p = a^{m+n+p}$, etc., where m, n, p are any positive whole numbers.

The exponent of a letter in a product is equal to the sum of the exponents of that letter in the factors multiplied.

69. The preceding rules enable us to immediately write the product of any number of monomials.

Ex. 1. $-3ab \times 5bc = -15ab^2c.$

Ex. 2. $4a^2b \times -3bc \times -5ac = 60a^3b^2c^2.$

Ex. 3. $-3x^2 \times 2xy \times -4ax \times -5by = -120abx^4y^2.$

In forming such products, three things require attention:

1. **The Sign.** This is + for positive factors and for an even number of negative factors; − for an odd number of negative factors.

2. **The Coefficient.** This is the product of the numerical factors formed as in arithmetic without regard to signs.

3. **The Literal Factors.** These consist of all the letters which occur, each letter having for exponent the sum of its exponents in the several expressions to be multiplied.

EXERCISE X

Multiply

1.	3	5	-3	-11	-8
	$\underline{-5}$	$\underline{-3}$	$\underline{-5}$	$\underline{7}$	$\underline{-5}$
2.	$3x$	$-2ab$	$4xy$	$-3a^2b$	$5ax$
	$\underline{-5x}$	$\underline{3ac}$	$\underline{-5x}$	$\underline{-2ab^2}$	$\underline{a^2b}$
3.	x^3y	$13x^2yz$	$-x^2y^3$	$8m^2n$	$7a^2b^2$
	$\underline{-y}$	$\underline{7y^2z}$	$\underline{-xy}$	$\underline{-3nx}$	$\underline{-5abc}$

4. $2a^2bc \times -3ab^2c \times 5abc^2.$

5. $5ab \times -3bx \times 2ax \times -4abx.$

6. Find the values of $(-5)^2$, $(-1)^2 \times 8$, $(-3)^3 \times -2.$

7. Find the values of $2^3 + (-2)^3$, $2^4 + (-2)^4.$

8. Simplify $(3-4)(-5+2) - 2^2$, $(-1)^3(5-7) - 3(-1).$

9. Simplify $(3a^3)^2$, $(2a^2)^3$, $(-2ab^2)^4$, $(-3a^2b^3)^5.$

10. If $a=2$, $b=-5$, find the values of

$$a+b,\ a-b,\ ab,\ ab(a-b),\ a^2+b^2.$$

11. If $a=2$, $b=-3$, $c=5$, find the value of

$$ab(a-b)+bc(b-c)+ca(c-a).$$

MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL

70. If either of two factors be multiplied by any number their product will be multiplied by the same number.

Thus $\quad 3\times 5=15$; $6\times 5=30$ and $3\times 10=30$;

that is, when either the 3 or the 5 is doubled their product 15 is also doubled.

71. If both of two addends be multiplied by any number their sum will also be multiplied by the same number.

Thus $\quad 3+5=8$; $6+10=16$;

that is, when both the 3 and the 5 are doubled their sum, 8, is also doubled.

72. The very important principles of Arts. 70, 71 are made evident to the eye by the following diagrams :

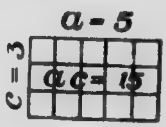


Fig. 1.

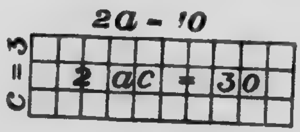


Fig. 2.

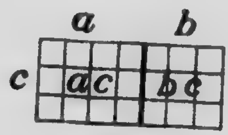


Fig. 3.

73. The area of a rectangle is the product of two factors, the length and the breadth. In Fig. 2 the length is double the length in Fig. 1, and its area is evidently also double, the width being the same in each. This illustrates the truth expressed in Art. 70.

In Fig. 3 the length is $a+b$, the width is c, and consequently the area is $c(a+b)$. If we now divide this rectangle into two rectangles whose lengths are a and b, their areas are ac and bc respectively.

Therefore $$c(a+b)=ac+bc,$$

which illustrates the truth of Art. 71.

74. The principle of Art. 71 may be extended to any number of addends, each of which may be positive or negative and consist of any number of factors. The multiplier may also contain two or more factors and be either positive or negative, *i.e.*, the multiplicand may be any **polynomial** and the multiplier any **monomial.** This gives us the following

Rule: *To multiply a polynomial by a monomial we multiply each term in succession and connect the partial products by the proper signs.*

Ex. 1. Multiply $3a^2-4ab+5b^2$ by $2a$.

Arrange thus
$$\begin{array}{l} 3a^2-4ab+5b^2 \\ 2a \\ \hline 6a^3-8a^2b+10ab^2 \end{array}$$

Begin at the left and work towards the right.

Ex. 2. Simplify $3a(2a^2+ab-2b^2)-5b(a^2-ab+3b^2)$.

This example consists of two multiplications similar to the preceding, followed by the addition of like terms in the two products.

$$\begin{array}{lrl} \text{Now} & 3a(2a^2+ab-2b^2) = & 6a^3+3a^2b-6ab^2 \\ \text{and} & -5b\ (a^2-ab+3b^2) = & -5a^2b+5ab^2-15b^3 \\ \hline \text{Adding, we get} & & 6a^3-2a^2b-ab^2-15b^3 \end{array}$$

the required result in its simplest form.

It will be observed that the negative sign connecting the two expressions was taken with the $5b$ as part of the multiplier, and the two products were then *added*. We might have taken $+5b$ as the multiplier and then the second product would have been *subtracted* from the first. The method given is usually the better one to follow.

EXERCISE XI

Multiply

1. $x^3 - 2x + 3$ by $2x$.
2. $3x^2 + 4x - 2$ by $-3x$.
3. $x^2 - 2xy + y^2$ by $2y$.
4. $2a^2 - 3ab + b^2$ by $-4ab$.
5. $1 - 2x + 3x^2$ by $-x$.
6. $xy + yz - xz$ by xyz.
7. $7a^2x - 2aby - 3xy^2 + 4b^2y$ by $-3abx$.
8. $2a^2 - 3b^2 - c^2 - 2ac - 4bc + 2ab$ by $-3abc$.
9. $1 - a + b - ac + bc - abc$ by a^2b.

Simplify

10. $3x(2x^2 - 5x + 6) + 2x(x^2 + 2x - 3)$.
11. $2x(x^2 - 2x + 3) - 5x(x^2 + 3x - 1) + 3x^2 - 4$.
12. $2a(a - b) + 3b(2a - 3b) - 2(a^2 - ab + 2b^2)$.
13. $a(2a^2 - 3ab - b^2) - 2b(a^2 - ab + 2b^2) - 3ab(2a - 3b)$.
14. $3(a - b + 2c) - 2(2a + 3b + 5c) + 5(b - 2c + 3a)$.
15. $a(a + b - c) - b(b + c - a) + c(c + a - b) - (a^2 + 2ab - b^2)$.
16. $2x\{3x - 2(x - 2y)\} - 3y\{2(x + 2y) - x\} + 5xy$.
17. $(px + qy) + (x + y) + (p - 1)x - (q + 1)y$.
18. $(a + b)x + (b + c)y - \{(a - b)x - (b - c)y\}$.
19. $(m + n)x + (m - n)y - m(x + y) - n(x - y)$.
20. $(a - b)x + (b - c)y + (c - a)z - a(x - y) - b(y - z)$
$- c(z - x) + bx + cy + az$.

MULTIPLICATION OF POLYNOMIALS BY POLYNOMIALS

75. In Art. 73 we have shown that $c(a+b)=ac+cb$, in which $a+b$ is multiplied by c. If now we take $a+b$ for the multiplier, the product in either case being the area of the rectangle, must remain unchanged.

That is $\qquad (a+b)c=ac+bc.$

Similarly $\qquad (a-b)c=ac-bc,$

as may easily be shown by a similar diagram.

If now we replace c by an expression of two addends $c+d$ or $c-d$, we get the following:

$$\begin{aligned}(a+b)(c+d)&=a(c+d)+b(c+d)\\&=ac+ad+bc+bd\end{aligned}$$

and

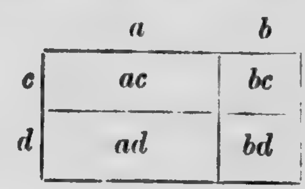
$$\begin{aligned}(a-b)(c-d)&=a(c-d)-b(c-d)\\&=ac-ad-bc+bd,\end{aligned}$$

which gives the rule for the multiplication of polynomials.

To multiply a polynomial by a polynomial.

Multiply each term of the multiplicand by each term of the multiplier and connect the partial products by the proper signs.

76. The process of the preceding Art. may be made evident to the eye by drawing a rectangle whose length is $a+b$ and width $c+d$ and dividing into four smaller rectangles as in the figure.



The area of the rectangle taken as a whole is

$$(a+b)(c+d).$$

The sum of the areas of the several parts is

$$ac+ad+bc+bd.$$

The two expressions must therefore be equal.

77. Two different classes of polynomials are of frequent occurrence, those whose terms consist of successive powers of one letter, and those in which two letters occur, the sum of whose exponents in each term is constant. Before multiplying such expressions they should be arranged so that the exponents of one of the letters in the successive terms will be either in descending or in ascending order of magnitude, as in the following examples.

Ex. 1. Multiply $2x^3 + x^2 - 3x - 4$ by $3x^2 - 2x + 1$.

$$\begin{array}{lrrrrrr}
\text{Arrange thus} & 2x^3 & +x^2 & -3x & -4 & & \\
 & 3x^2 & -2x & +1 & & & \\
\hline
 & 6x^5 & +3x^4 & -9x^3 & -12x^2 & & \\
 & & -4x^4 & -2x^3 & +6x^2 & +8x & \\
 & & & 2x^3 & +x^2 & -3x & -4 \\
\hline
\text{Product} & 6x^5 & -x^4 & -9x^3 & -5x^2 & +5x & -4
\end{array}$$

Having arranged the terms with their exponents in order of magnitude, we begin at the left, multiplying each term of the multiplicand by $3x^2$; then by $-2x$ placing each term of the product in the second line under the like term in the first line; similarly with the third line; finally the sum of the three lines is the product required.

Ex. 2. Multiply $a^2 + ab + b^2$ by $a^2 - ab + b^2$.

$$\begin{array}{lrrrrr}
\text{Arrange as before} & a^2 & +ab & +b^2 & & \\
 & a^2 & -ab & +b^2 & & \\
\hline
 & a^4 & +a^3b & +a^2b^2 & & \\
 & & -a^3b & -a^2b^2 & -ab^3 & \\
 & & & a^2b^2 & +ab^3 & +b^4 \\
\hline
\text{Product} & a^4 & & +a^2b^2 & & +b^4
\end{array}$$

The two examples here given belong to the classes specified at the beginning of this Art. Such examples may always be worked by the brief method shown in the next example.

Ex. 3. Multiply $x^4 - 2x^3 + 3x^2 + 5$ by $x^3 + 2x^2 - 3$.

Arrange the coefficients of multiplicand and multiplier in the usual order for multiplication, but note that in each expression there is one term wanting, and place a cypher in the vacant place. The work of multiplication may then be arranged as follows :

$$
\begin{array}{rrrrrrrr}
1 & -2 & +3 & +0 & +5 & & & \\
1 & +2 & +0 & -3 & & & & \\
\hline
1 & -2 & +3 & +0 & +5 & & & \\
 & 2 & -4 & +6 & +0 & +10 & & \\
 & & & -3 & +6 & -9 & -0 & -15 \\
\hline
1 & +0 & -1 & +3 & +11 & +1 & -0 & -15
\end{array}
$$

Result $\qquad x^7 - x^5 + 3x^4 + 11x^3 + x^2 - 15.$

The student will observe that the purpose of the cyphers is to keep the other coefficients in their proper columns. The highest exponent, 7, is obtained by taking the sum of the highest exponents in multiplicand and multiplier. Polynomials of either class described in this Art. can be multiplied by this method, which is called "multiplying by detached coefficients."

EXERCISE XII

Multiply

1. $2x^2 - x + 3$ by $3x - 2$.
2. $x^2 + 2x - 3$ by $2x - 1$.
3. $4x^2 - 5x - 2$ by $5x + 3$.
4. $3x^2 + x - 5$ by $-x + 2$.
5. $x^2 + 2x + 4$ by $x - 2$.
6. $a^2 - a + 1$ by $a + 1$.
7. $a^2 + ab + b^2$ by $a - b$.
8. $a^2 - ab + b^2$ by $a + b$.
9. $a^2 - 2a + 3$ by $a^2 + 2a - 3$.
10. $2a^2 - 5ab + 3b^2$ by $2a^2 + 5ab + 3b^2$.
11. $2x^3 - x^2 + 3x - 1$ by $3x^2 + x - 2$.

12. $3x^3 - 7x^2 - 4x + 5$ by $2x^2 + 3x - 1$.

13. $1 - 2x + 3x^2 + 4x^4$ by $1 + 2x - 3x^2$.

14. $2 - x^3 + 3x^2 - 4x + x^4$ by $1 - 2x + x^2$.

15. $4x - 2x^2 + 3x^3 - 1$ by $2x - x^2 + 3$.

16. $x^5 - x^4 + x^3 - x^2 + x - 1$ by $1 + x$.

17. $a^4 + 2a^3 + 3a^2 + 2a + 1$ by $a^4 - 2a^3 + 3a^2 - 2a + 1$.

18. $x^5 - 3x^4 + 2x^2 - 5x + 7$ by $x^3 - 2x + 3$.

19. $x^2 + xy + y^2 - x + y + 1$ by $x - y + 1$.

20. $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.

21. $a^2 + 2ab + b^2 - c^2$ by $c^2 - a^2 + 2ab - b^2$.

22. $x^2 + 4y^2 + z^2 + 2xy + 2yz - xz$ by $x - 2y + z$.

23. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ by
$a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.

24. $x^8 + x^7y - x^5y^3 - x^4y^4 - x^3y^5 + xy^7 + y^8$ by
$x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

Simplify

25. $(x-1)(2x+3) + (2x-1)(x+3) - (3x+2)(x-5)$.

26. $2(1-x)(1+x+x^2) + 3(1+x)(1-x+x^2)$
$- x(x+1)(x-1)$.

27. $(x+1)(x+2)(x-3) - (x-1)(x-2)(x+3)$.

28. $(a+b)(a-b) - (b-c)(b+c) - (c-a)(c+a)$.

29. $(a-b)(a+b-c) + (b-c)(b+c-a) + (c-a)(c+a-b)$.

30. $a(a+b)^2 - b(a-b)^2 - ab(a+3b)$. ?

31. $(a^2+ab+b^2)(a^2-ab+b^2) - a^2(a+b)(a-b)$.

32. $(a-2)(a+2)(a^2+4) + (b-2)(b+2)(b^2+4)$
$- (a^2+b^2)(a^2-b^2)$.

DIVISION

78. The meaning of Division of numbers and the rules for performing it are derived directly from multiplication, as the following simple examples will show:

$$\frac{20}{4}=5, \qquad \frac{-20}{4}=-5, \qquad \frac{20}{-4}=-5, \qquad \frac{-20}{-4}=5.$$

The first example is that of arithmetic, in which we seek a number which multiplied by 4 will make 20; the multiplication table gives the required number 5. In the next example we seek a multiplier of 4 which gives -20; as before, the absolute number is 5, but the rule of signs in multiplication requires it to be negative, viz., -5. Similarly for the other two examples, observe that the number to be divided is the product of two factors, one of which is the number by which we divide, and the remaining factor is the quotient.

79. Definition. When the product of two factors and one of them are given, the process of finding the remaining factor is called **Division.** The given product is called the **Dividend,** the given factor is called the **Divisor** and the factor to be found is called the **Quotient.**

Since $(+a)\times(+b)=+ab \quad \therefore \quad \dfrac{+ab}{+b}=+a$ Art. 65.

$$(-a)\times(+b)=-ab \quad \therefore \quad \frac{-ab}{+b}=-a$$

$$(+a)\times(-b)=-ab \quad \therefore \quad \frac{-ab}{-b}=+a$$

$$(-a)\times(-b)=+ab \quad \therefore \quad \frac{+ab}{-b}=-a$$

That is, the sign of the quotient of two numbers is + when both numbers have the same sign, and $-$ when

they have different signs. From this we have, as in multiplication, the

Rule of Signs. *Like signs give* $+$, *unlike signs give* $-$.

80. In the previous Art. the dividend is represented by just two factors and the divisor by one, but both dividend and divisor may have any number of factors. In such cases each of the factors of the divisor must be removed from the dividend; the remaining factors constitute the quotient.

Thus $\qquad \dfrac{abcd}{ab}=cd, \qquad \dfrac{10xy(x-y)}{5(x-y)}=2xy$, etc.

81. The quotient of two powers of the same factor is formed thus:

Since $\qquad a^5=aaaaa$, and $a^2=aa$,

Therefore $\quad \dfrac{a^5}{a^2}=\dfrac{aaaaa}{aa}=aaa=a^3=a^{5-2}$.

Similarly $\quad \dfrac{a^m}{a^n}=a^{m-n}$, in which m and n may be any positive whole numbers.

82. Index law in division. *The exponent of a letter in a quotient is obtained by subtracting the exponent of the divisor from the exponent of the dividend.*

83. The close analogy between division and subtraction should be carefully observed.

To remove an addend from an algebraic expression is subtraction; to remove a factor is division. Thus removing a from $a+b$ is subtraction; removing a from ab is division.

Again, to change $5a$ to $3a$ is to subtract $2a$; to change a^5 to a^3 is to divide by a^2.

The difference of two expressions remains unchanged if a new addend be introduced or removed from each; the quotient remains unchanged if a new factor be introduced or removed from each.

Thus $\quad (a+b)-a=b$ and $(a+b+x)-(a+x)=b$

$$\frac{ab}{a}=b \text{ and } \frac{abx}{ax}=b, \text{ etc.}$$

84. Division of polynomials by a monomial. From Art. 74, it is evident that a polynomial is divided by a monomial by dividing each of its terms in succession and connecting the partial quotients by the proper signs.

Ex. 1. $\dfrac{10abc}{-5a}=-bc, \quad \dfrac{-24a^3xy^2}{-3ay^2}=8a^2x, \quad \dfrac{(a+b)^5}{(a+b)^2}=(a+b)^3.$

Ex. 2. $\dfrac{4a^3b-6a^2b^2-10ab^3}{2ab}=2a^2-3ab-5b^2.$

EXERCISE XIII

1. $\dfrac{20}{-5}, \quad \dfrac{-20}{5}, \quad \dfrac{-20}{-5}, \quad \dfrac{-64}{-16}.$
2. $\dfrac{(-6)^3}{(-2)^2}, \quad \dfrac{-750}{(-5)^2}, \quad \dfrac{75}{(-3)(-5)}, \quad \dfrac{84}{(-1)^{34}}.$
3. $\dfrac{8a^2b^3}{2ab}, \quad \dfrac{12a^3b^2c}{-3abc}, \quad \dfrac{-14x^2y}{-2xy}, \quad \dfrac{-18m^2n^5}{3m^2n}.$
4. $\dfrac{84x^5y^{10}}{-4xy^5}, \quad \dfrac{-40a^2x^3y}{10axy}, \quad \dfrac{75b^4c^5}{-15bc^5}, \quad \dfrac{-72a^5b^2c^3}{12a^2c}.$
5. $6a^2b\times 5ab^2$ by $10a^2b^2$; $2ax\times -3by$ by $-6xy$.
6. $3xy\times -5yz\times -6xz$ by $2x^2\times -3y$.
7. $4x^3-6x^2+8x$ by $2x$.
8. $9y^3+12y^2-6y$ by $-3y$.
9. $8a^4-16a^3b+24a^2b^2$ by $8a^2$.

10. $25a^3b^2 - 50a^2b^3 - 100ab^2c$ by $-25ab^2$.

11. $-3x^4y + 5x^3y^2 - 6x^2y^3 - xy^4$ by $-xy$.

12. $-49x^5yz^3 + 63x^4y^2z - 56x^3y^3z^2$ by $-7x^3yz$.

13. $6(a+b)^2 - 8(a+b)^3 + 10(a+b)^4$ by $2(a+b)^2$.

14. $x^2y(x-y) - yz(x-y)^2 + y^3(x-y)$ by $y(x-y)$.

15. $(3ab^2 - 3a^2b + 6ab^2)(2a^2b + 2ab^2)$ by $6a^2b^2$.

DIVISION OF POLYNOMIALS BY POLYNOMIALS

85. The rule for the division of one polynomial by another is obtained by closely observing the mode of multiplying one polynomial by another and then reversing the process.

Suppose the divisor to be $3x^2 - 5x + 7$

and the quotient $2x - 4$

$$\begin{array}{rrrr} 6x^3 & -10x^2 & +14x & \\ & -12x^2 & +20x & -28 \\ \hline 6x^3 & -22x^2 & +34x & -28 \end{array}$$

then the dividend is $6x^3 - 22x^2 + 34x - 28$

Now observe:

1. The first term of the dividend, $6x^3$, is the product of the first term of the divisor, $3x^2$, and the first term of the quotient.

2. Therefore, the first term of the quotient, $2x$, is to be obtained by dividing the first term of the divisor, $3x^2$, into the first term of the dividend, $6x^3$.

86. The dividend is the sum of the products of the divisor by the several terms of the quotient. Therefore, if from the dividend we subtract the product of the divisor by $2x$, the first term of the quotient, the remainder must be the product of the divisor and the remaining term of the quotient (or the sum of such products when more than one term of the quotient remains to be found).

The work may therefore be arranged as follows:

$$\begin{array}{r|l|l} 3x^2-5x+7 & 6x^3-22x^2+34x-28 & 2x-4 \\ & 6x^3-10x^2+14x & \\ \hline & \quad -12x^2+20x-28 & \\ & \quad -12x^2+20x-28 & \end{array}$$

87. To divide a polynomial by a polynomial.

Arrange the terms of divisor and dividend both in descending or both in ascending powers of a common letter.

Divide the first term of the dividend by the first term of the divisor; the result will be the first term of the quotient.

Multiply each term of the divisor by the first term of the quotient and subtract the product from the dividend.

If there be a remainder, consider it a new dividend and proceed as before.

88. It is essential that the terms in the several remainders be kept in the same order with regard to the exponents of the letter of reference. If a remainder occurs in which the highest exponent of the letter of reference is lower than the highest exponent of that letter in the divisor, the division cannot be exactly performed. Such examples will be further considered in the chapter on Fractions.

The following are additional examples:

Ex. 1. Divide $x^3-9x^2+23x-30$ by $x-6$.

$$\begin{array}{r|l|l} x-6 & x^3-9x^2+23x-30 & x^2-3x+5 \\ & x^3-6x^2 & \\ \hline & \quad -3x^2+23x & \\ & \quad -3x^2+18x & \\ \hline & \qquad\qquad 5x-30 & \\ & \qquad\qquad 5x-30 & \end{array}$$

Ex. 2. Divide $a^3 + b^3$ by $a + b$.

$$
\begin{array}{l|l|l}
a+b & a^3 + b^3 & a^2 - ab + b^2 \\
 & a^3 + a^2b & \\ \hline
 & -a^2b + b^3 & \\
 & -a^2b - ab^2 & \\ \hline
 & \quad ab^2 + b^3 & \\
 & \quad ab^2 + b^3 & \\ \hline
\end{array}
$$

Ex. 3. Divide $a^3 - b^3$ by $a^2 + ab + b^2$.

$$
\begin{array}{l|l|l}
a^2 + ab + b^2 & a^3 \qquad\quad - b^3 & a - b \\
 & a^3 + a^2b + ab^2 & \\ \hline
 & \quad - a^2b - ab^2 - b^3 & \\
 & \quad - a^2b - ab^2 - b^3 & \\ \hline
\end{array}
$$

Ex. 4. Divide $1 - 4x^2 + 16x^3 - x^4 - 12x^5$ by $1 + 2x - 3x^2$.

$$
\begin{array}{l|l|l}
1 + 2x - 3x^2 & 1 \qquad - 4x^2 + 16x^3 - x^4 - 12x^5 & 1 - 2x + 3x^2 + 4x^3 \\
 & 1 + 2x - 3x^2 & \\ \hline
 & -2x - x^2 + 16x^3 & \\
 & -2x - 4x^2 + 6x^3 & \\ \hline
 & \qquad 3x^2 + 10x^3 - x^4 & \\
 & \qquad 3x^2 + 6x^3 - 9x^4 & \\ \hline
 & \qquad\qquad 4x^3 + 8x^4 - 12x^5 & \\
 & \qquad\qquad 4x^3 + 8x^4 - 12x^5 & \\ \hline
\end{array}
$$

When a term of the regular series in the dividend is wanting, as in Exs. 3 and 4, it is convenient to leave a vacant space in order to permit like terms to be placed in the same column.

If both divisor and dividend are not already in their simplest forms, as in the preceding examples, they must be simplified by performing any indicated multiplications and collecting like terms before attempting to perform the division.

EXERCISE XIV

Divide

1. $x^2+10x+21$ by $x+3$.
2. $x^2-11x+24$ by $x-8$.
3. x^2-x-56 by $x+7$.
4. x^2+x-90 by $x+10$.
5. $4x^2-9$ by $2x-3$.
6. x^3-7x+6 by $x-2$.
7. a^2-b^2 by $a-b$.
8. a^3-b^3 by $a-b$.
9. a^3+b^3 by $a+b$.
10. a^3-b^3 by a^2+ab+b^2.
11. x^3-7x-6 by $x-3$.
12. $4x^3+5x+21$ by $2x+3$.
13. $2x^3+7x^2+5x+100$ by $2x^2-3x+20$.
14. $a^5-5a^3+7a^2+6a+1$ by a^2+3a+1.
15. $3a^4-5a^3b+a^2b^2+13ab^3+4b^4$ by $a^2-3ab+4b^2$.
16. $4x^5+7x^3-6x-12x^4+5x^2+3$ by $2x^2+3-x$.
17. $19x^4-x^2+10+3x^6-11x^5-13x^3$ by $3+x^2-2x$.
18. $a^4+a^2b^2+b^4$ by a^2+ab+b^2.
19. $a^8+a^4b^4+b^8$ by $a^4-a^2b^2+b^4$.
20. $a^{15}+b^{15}$ by a^5+b^5 and by a^3+b^3.
21. $a^{15}-b^{15}$ by $a^{10}+a^5b^5+b^{10}$
and by $a^{12}+a^9b^3+a^6b^6+a^3b^9+b^{12}$.
22. x^6-2x^3+1 by x^2-2x+1 and by x^4-x^3-x+1.
23. x^7-x by x^3-x and by $x^4-2x^3+2x^2-x$.
24. $a^5-b^5+a^2b^3-a^3b^2$ by $a^3-b^3-2a^2b+2ab^2$.
25. $x^5-y^5-x^3y^2+x^2y^3$ by x^2+y^2+2xy.
26. $8a^6-b^6+21a^3b^3-24a^5b$ by $3ab-a^2-b^2$.
27. $x^4-px^3+px^2-p^2x$ by $x-p$.
28. $x^4+2mx^3+m^2x^2-n^2$ by x^2+mx+n.
29. $a^3x^3+a^2bx^2y-ab^2xy^2-b^3y^3$ by $ax-by$.
30. $6a^4x-17a^3x^2+44a^2x^3-48ax^4$ by $2a^2-3ax$.
31. Divide the product of x^2+x-2 and $2x^2-7x+6$
by $2x^2+x-6$.

32. Divide $x^7 - 64x$ by $x(x+2)+4$.

33. Divide $x^2(x^3-5)+5x-1$ by $x(x-2)+1$.

34. Divide $a(a+b)^2 - b(a-b)^2 + 2b(a^2+b^2)$ by $a+b$.

35. Divide $(a+b+1)(a+b-2)-10$ by $a+b-4$.

36. Divide $(a^2+ab+b^2)(a^2-ab+b^2)+a^2b^2$
by $(a+b)^2-2ab$.

37. Divide $(a+b)(a^2-ab+b^2)+3ab-1$ by $a+b-1$.

38. Divide a^6-6a+5 by $(a^2+a+1)^2+2(a+2)$.

39. Divide $(x+1)^2(x^3-x+5)+(2x-1)(x^2-2x+3)$
by $(x-1)^2+1$.

40. Divide $(x-1)^2(x^3-x^2+1)-(3x-2)(x^2+x-1)$
$+18x+1$ by $(x+1)^2+2$.

LITERAL COEFFICIENTS

89. Meaning of Coefficients extended. The definition of a coefficient in Art. 11 tacitly implies that it is a whole number and expressed in figures, and coefficients of this kind are the only ones thus far brought into use. The definition is there given in simple form for beginners and expresses only a part of the truth. A coefficient is a factor or multiplier, and as such may be any algebraical expression, integral or fractional, and expressed by letters or figures.

Thus in ax, by, $(a+b)x$, $(a-b)y$, the literal factors a, b, $a+b$, $a-b$ may be considered literal coefficients of the factors which follow them.

It is equally true that x and y might be considered coefficients of the factors which precede them.

90. Use of Brackets. By the aid of the preceding Art. the sum of two or more unlike terms, having at least one literal factor in common, may be expressed as a single

term by considering the unlike factors as literal coefficients and enclosing their sum in a bracket.

For just as $\quad 2x + 3x = (2+3)x = 5x,$

so $\quad ax + bx = (a+b)x,$

the only distinction being that in the first example we have a single symbol, 5, to take the place of $2+3$, but we have no single symbol to stand in place of $a+b$, and consequently the expression $(a+b)x$ cannot be further simplified.

The following are additional examples:

Ex. 1.

To	ax	$(a-b)y$	ay	$(a+b+c)y$
add	$-bx$	$2by$	y	$(b-2c)y$
Sum	$(a-b)x$	$(a+b)y$	$(a+1)y$	$(a+2b-c)y$

Ex. 2.

From	$(a+b)m-(a+b)n+(a-b)p-(a-b)q$
take	$(a-b)m-(a-c)n-(b-c)p+(b+c)q$
Result	$2bm-(b+c)n+(a-c)p-(a+c)q$

91. In subtracting terms with literal coefficients, we follow the ordinary rule, but we have a choice of two ways of applying it. When the signs preceding the brackets are alike, as in the case of the coefficients of m and n, we change the sign of each term *within* the bracket and add, placing the common sign before the result. The coefficients of p and q have unlike signs preceding the bracket, and in such cases it is best to change the sign which precedes the bracket of the lower terms and add the terms within the brackets as they stand.

92. The following are important examples in multiplication and division, in which the terms containing the same power of x are combined:

Ex. 1. Find the continued product of

$$x+a,\ x+b \text{ and } x+c.$$

$$\begin{array}{l}
x+a \\
x+b \\
\hline
x^2+ax \\
\quad\ +bx+ab \\
\hline
x^2+(a+b)x+ab \\
x+c \\
\hline
x^3+(a+b)x^2+abx \\
\qquad\ \ cx^2+(ac+bc)x+abc \\
\hline
x^3+(a+b+c)x^2+(ab+bc+ac)x+abc
\end{array}$$

Ex. 2. Divide $x^3-(a+b+c)x^2+(ab+bc+ac)x-abc$

by $x-a$.

$$\begin{array}{r|l}
x-a \;\; x^3-(a+b+c)x^2+(ab+bc+ac)x-abc & x^2-(b+c)x+bc \\
x^3-ax^2 \qquad\qquad\qquad\qquad\quad & \\
\hline
-(b+c)x^2+(ab+bc+ac)x \qquad & \\
-(b+c)x^2+(ab+ac)x \qquad & \\
\hline
bcx-abc & \\
bcx-abc & \\
\hline
\end{array}$$

EXERCISE XV

Collect the coefficients of x and y in Exs. 1–8.

1. [illegible] $+by+$ [illegible] $-px+qy$.
2. [illegible] $+$ [illegible] $+7y$.
3. n [illegible] $ny-nx$.
4. a^2x+ [illegible] $-2a$ [illegible] $x-b^2y$.
5. $ax+by+bx+ay-(a+1)x-(b-1)y+2x+3y$.
6. $(a-2b)x+(b-3c)y+2bx-by+(n-a)x-(m-3c)y$.
7. $nx+ny+(m-n)x+(m-n)y-2mx-my$.
8. $(a-b)x+(b-c)y+(b-c)x+(c-a)y+(a-c)x$
$-(a-b)y$.

9. From $ax^2+bxy-cy^2$ take $px^2-qxy+ry^2$.

10. From $(a-b)x^2+(b+c)xy-(c-a)y^2$
take $(a-c)x^2-(a-c)xy+(a-b)y^2$.

11. From the sum of
$$(a-b)x+(b-c)y+(c-a)z \text{ and } bx-cy+z$$
take the sum of $(b-c)x+y+z$ and $(c+1)x-2cy+(c-1)z$.

Arrange the products according to the powers of x in Exs. 12–20.

12. $(x+a)(x+b)$.

13. $(x-a)(x-b)$.

14. $(x+a)(x-b)$.

15. $(x-a)(x+b)$.

16. $(x+a)(x+b)(x+c)$.

17. $(x-a)(x-b)(x-c)$.

18. $(x+a)(x-b)(x+c)$.

19. $(x-a)(x+b)(x-c)$.

20. $(x-a)(x+b)+(x-b)(x+c)+(x-c)(x+a)$.

21. Divide $a^2-b^2-c^2+2bc$ by $a-b+c$.

22. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

23. Divide $8a^3-b^3+c^3+6abc$ by $2a-b+c$.

24. Divide $x^2+xy+2xz-2y^2+7yz-3z^2$ by $x-y+3z$.

25. Divide $x^2-(a+b)x+ab$ by $x-a$.

26. Divide $x^2-(a+b+c)x+a(b+c)$ by $x-(b+c)$.

27. Divide $x^3-(a+m)x^2+(am+mn)x-amn$ by $x-a$.

28. Divide $x^3+(a+b+c)x^2+(ab+bc+ac)x+abc$
by $x+a$ and by $x+b$.

29. Divide $x^3-(a+b+c)x^2+(ab+bc+ac)x-abc$
by $x-b$ and by $x-c$.

30. Divide $x^3+(a+b-c)x^2+(ab-bc-ac)x-abc$
by $x^2+(a-c)x-ac$.

31. Divide $a^2(b+c)+a(b^2+c^2)-2bc(b+c)$ by $a+b+c$.

32. Divide $2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4$
by $2ab-a^2-b^2+c^2$ and by $2bc-b^2-c^2+a^2$.

CHAPTER IV

SIMPLE EQUATIONS: ONE UNKNOWN QUANTITY

(ELEMENTARY)

93. Definition. An Equation is a statement that two algebraical expressions are equal, *i.e.*, they represent the same number.

Thus $2x+3x=5x$, $2(x-5)=2x-10$, $2x-7=x+3$ are equations.

94. Sides of an equation. The expressions on opposite sides of the sign of equality, $=$, are called the **sides,** or **members,** of the equation.

95. Two expressions may represent the same number in different ways.

1. They may represent the same number for all values of the letters which they contain.

Thus $2x+3x=5x$, and $2(x-5)=2x-10$, whatever value may be assigned to x, *i.e.*, x represents a general number.

2. They may represent the same number for only one or more particular values of the letter.

Thus $2x-7=x+3$ only when $x=10$, and then each side of the equation becomes 13. For all other values of x the two expressions are unequal, *i.e.*, x represents a particular number.

96. Identities. Two expressions which are equal for all values of the letters contained form an **identity** or, as it is sometimes called, an **identical equation**.

An identity always consists of two expressions, of which the one may be derived from the other by the ordinary rules of algebra.

97. Conditional equations. Two expressions which are equal for one or more particular values form a **conditional equation.** The word "conditional," however, is usually omitted and the name equation alone used. It is with this class of equation that elementary algebra is chiefly concerned.

98. Unknown quantity. The letter to which a particular value must be given to make the sides equal, is called the **unknown quantity.**

99. Solving an equation. To solve an equation is to find the value of the unknown quantity which makes the two sides equal. This value is called a **Root** of the equation and is said to satisfy the equation, *i.e.*, to make the two sides equal.

100. Simple equation. A simple equation is one which does not contain the square or any higher power of the unknown quantity. Such equations have but one root.

Thus $3x - 1 = 2(x+3)$ is a simple equation and is satisfied by but one value of x, viz. $x = 7$.

But $x^2 - x = 6$ is not a simple equation. It is satisfied when $x = 3$ or -2, but for no other value.

101. Axioms used in solving equations.

1. If equals be added to equals the sums are equal.
2. If equals be taken from equals the remainders are equal.
3. If equals be multiplied by equals the products are equal.

4. If equals be divided by equals (not zero) the quotients are equal.

102. Illustrations. If two rods have the same length they will still be equal if an inch be added to or subtracted from each; also if each be multiplied or divided by the same number. A pair of scales, having equal arms, balances when the weights in the two scale-pans are equal. If the same additional weight be placed in, or removed from each pan, the balance will not be destroyed.

103. *Ex.* Solve the equation $5x-7=3x+11$ and verify the result.

SOLUTION

The given equation is	$5x-7=3x+11$	
Add 7 to each side	$5x=3x+11+7$	(Ax. 1.)
Subtract $3x$ from each	$5x-3x=11+7$	(Ax. 2.)
Combine like terms	$2x=18$	
Divide by coefficient of x	$x=9$	(Ax. 4.)

VERIFICATION

When $x=9$, $5x-7=5(9)-7=45-7=38$
and $3x+11=3(9)+11=27+11=38$,
which proves the solution correct.

104. In the preceding solution observe:

1. Adding 7 to each side caused the -7 to disappear from the first side of the equation, and $+7$ to appear instead upon the opposite side.

2. Subtracting $3x$ from each side caused $+3x$ to disappear from the second side and to reappear as $-3x$ on the first side.

3. The object in making these changes was to collect all the terms containing x on one side and all other terms on the other side of the equation.

These observations furnish the mode of solution of a simple equation and the reasoning upon which it is founded.

105. Transposing terms. *A term may be transposed from one side of an equation to the other, without destroying the equality, providing the sign of the term be changed.*

With this statement compare the first two statements of Art. 104.

106. Rule for solving. *Remove brackets, if any occur; transpose all terms containing the unknown quantity to the first side, and all remaining terms to the second side of the equation; combine like terms, and divide both sides by the coefficient of the unknown quantity.*

Ex. 1. Solve the equation $2x - 3(x-2) = 4 + 2(x-5)$.

Remove brackets $\quad 2x - 3x + 6 = 4 + 2x - 10$

Transpose terms $\quad 2x - 3x - 2x = 4 - 10 - 6$

Combine terms $\quad -3x = -12$

Divide by -3 $\quad x = 4$ (Ax. 4.)

Verification $\quad 2x - 3(x-2) = 2(4) - 3(2) = 8 - 6 = 2$

$\quad 4 + 2(x-5) = 4 + 2(-1) = 4 - 2 = 2$

Ex. 2. Solve equation $2(x+3)^2 - 13 = (x-1)^2 + (x-2)^2$.

Simplify each side $\quad 2x^2 + 12x + 5 = 2x^2 - 6x + 5$

Transpose $\quad 2x^2 - 2x^2 + 12x + 6x = 5 - 5$

Combine $\quad 18x = 0$

Divide by 18 $\quad x = 0$

Verification $\quad 2(x+3)^2 - 13 = 2(3)^2 - 13 = 5$

$\quad (x-1)^2 + (x-2)^2 = (-1)^2 + (-2)^2 = 5$

Ex. 3. Solve equation $a(x-a)+b(x-b)=2ab$.

Remove brackets $\qquad ax-a^2+bx-b^2=2ab$

Transpose $\qquad ax+bx=a^2+2ab+b^2$

Combine terms in x $\qquad (a+b).=a^2+2ab+b^2$

Divide by $a+b$ $\qquad x=a+b$

Verification $\qquad a(a+b-a)+b(a+b-b)=ab+ab$

$$=2ab$$

EXERCISE XVI

107. Solve the following equations and verify each result.

1. $2x-5=x+7$.
2. $3x+4=x+22$.
3. $3(x-5)=2(x+1)+10$.
4. $2(x+7)=3(x-11)$.
5. $x-(5-2x)=4(1-x)+5$.
6. $2(x+2)+3(x-5)+1=0$.
7. $x-(4x-5)+3(x-7)=x$.
8. $3x+7-(5x-2)-11=0$.
9. $2(x-2)+3(x-3)-4(x-4)=0$.
10. $5x-2(7-3x)+4(2x-5)-(x+2)=0$.
11. $8-2(3x+5)-2(2-5x)=8x-(6-11x)$.
12. $x-7(4x-11)=14(x-5)-19(8-x)-61$.
13. $5x+6(x+1)-7(x+2)-8(x+3)-2(x+8)=0$.
14. $(x+3)(x+7)=(x+2)(x-12)$.
15. $(x-1)(x-9)=(x+2)(x-11)$.
16. $(x+8)(x-11)-2(x+3)(x-7)=x(4-x)+52$.
17. $(x-2)(7-x)+(x-5)(x+3)-2(x-1)+12=0$.
18. $(2x-3)(x+7)-(x-5)(2x+3)=x(x-2)-x^2$.
19. $(2x-7)(x+5)=(9-2x)(4-x)+229$.
20. $(x-1)(x-3)(x-5)=(x-2)(x-3)(x-4)$.

+ **21.** $(x-1)^2+(x-3)^2-2(x-5)^2=12.$

22. $(x-1)^3+(x-2)^3+(x-3)^3=3(x-1)(x-2)(x-3).$

23. $5(a+x)+3(a-x)=3x.$

24. $a(x-a)+b(x-b)=a(x+b)-a^2.$

25. $(x+a)(x-b)-(x-a)(x+b)=(a-b)x.$

26. $a(x-a)=b(x-b).$ **27.** $(a+b)x+(a-b)x=a^2.$

28. $(a+b)x-(a-b)x=b^2.$ **29.** $ax-bx=a+bx-2b.$

30. $a(x+a)+b(b-x)=2ab.$ **31.** $a(x+a)=b(x-b)+2a^2.$

32. $(x-a)^2-(x-a)(x-b)+(x-b)^2=x(x-a)+a^2.$

+ **33.** $x(a+b)+(a+b)^2=c(c-x).$

34. $x(x-a)+b(x-b)=(x-a)^2+2ab.$

35. $(a+b)(x-c)+(b+c)(x-a)=(c-a)(x+b).$

+ **36.** $a(x-a)+b(x-b)-c(x-c)=2ab.$

PROBLEMS

108. Algebra is extensively used in the solution of problems in which one or more numbers are to be found from their connection with other numbers already known. The mode of proceeding can best be learned from a few simple examples.

Ex. 1. A purse contains dimes and quarter-dollars, 15 more dimes than quarters, and 51 coins in all. Find the number of each.

Let $x=$ number of quarter-dollars,

then $x+15=$ " dimes.

Adding $x+(x+15)=$ total number of coins

$=51$, the number given.

Therefore $2x+15=51,$

from which $x=18$, the number of quarters,

and $x+15=33$, " " dimes.

Ex. 2. In the preceding example, if the total value of the coins had been given, \$7.80, instead of their number, the solution would have been as follows:

Let $x =$ number of quarter-dollars,
then $x + 15 =$ " dimes,
and $25x =$ " cents in 25 quarters,
" $10(x+15) =$ " " $x+15$ dimes.
Adding $25x + 10(x+15) =$ total number of cents
$= 780$ cents, the number given.

Therefore $35x + 150 = 780$,
from which $x = 18$, the number of quarters,
and $x + 15 = 33$, " " dimes.

Ex. 3. A father, 40 years of age, has 3 children of 10, 8 and 6 years of age. In how many years will the sum of the ages of his children be double his own age?

Let $x =$ the number of years required,
then $40 + x =$ father's age after x years
and $10 + x =$ eldest child's age
" $8 + x =$ second "
" $6 + x =$ youngest "
" $24 + 3x =$ sum of children's ages,
" $2(40 + x) =$ twice father's age.
Therefore $24 + 3x = 2(40 + x)$
solving $x = 36$, the number of years required.

Ex. 4. A yard of velvet is worth 50 cents more than a yard of silk; 10 yards of silk and 12 yards of velvet are together worth \$61. Find the value of a yard of each.

Let	$x =$	number of cents	for a	yard of	silk,
then	$x+50 =$	"	"	"	velvet;
"	$10x =$	"	"	10 yards of	silk;
"	$12(x+50) =$	"	"	12 "	velvet.

Adding $\quad 10x+12(x+50)=6100.$

Solving equation $\quad x=250,$

and $\quad x+50=300.$

The values are, therefore, \$2.50 and \$3 respectively.

109. No rules can be given for the solution of problems, but the following observations may be of some assistance as a general guide:

1. Let x stand for the number from which the other numbers connected with the problem can be most easily found.

2. Find from the problem two different expressions, each of which represents the same number. These will form the two sides of the equation.

3. When concrete quantities occur, they must all be expressed in units of the same denomination.

4. Be careful to specify clearly the units which x is used to count; x must stand for an *abstract* number.

EXERCISE XVII

1. Find the number which when multiplied by 3 and with 11 added to the product makes 47.

2. The sum of two numbers is 75 and their difference 17. Find the numbers.

3. The double of a certain number is greater by 3 than the number itself with 7 added. Find the number.

4. Two boys together have \$1.25 and one of them has 17 cents more than the other. How much has each?

5. Two boys had each the same number of cents. One of them lost 25 cents and the other earned 7 cents, and then the latter had twice as many as the former. How many had each at first?

6. The length of a room is 3 times its width. If it were 10 ft. shorter and 12 ft. wider it would be square. Find its width.

7. The length of a room is 8 ft. more than its width and its perimeter is 60 ft. Find its length.

8. *A* and *B* have each the same amount of money; *A* gains \$17 and *B* loses \$3 and now *A* has 3 times as much as *B*. How much had each at first?

X 9. *A* is three times as old as *B*, but in 10 years he will be only twice as old. How old is *A* at present?

10. Two boys have each the same number of marbles; one of them wins 24 from the other and now the former has 4 times as many as the latter. How many had each at first?

11. Divide 56 into three parts such that the first part may exceed the second part by 8 and the third part by 11.

12. Divide 75 into two parts such that 12 times one part may equal 13 times the other part.

13. At an election 875 votes were cast and the successful candidate had 29 majority. How many votes did each receive?

14. Divide 89 into three parts such that the first part may be less than the second by 5 and less than the third by 12.

15. Divide \$7.64 between *A*, *B* and *C*, giving *A* 5 cents less than *B*, and *C* 2 cents more than *A* and *B* together.

16. Two men have each the same amount of money. One of them gives the other \$50 and now the latter has 3 times as much as the former. How much had each at first?

17. The ages of two men differ by 20 years, and 15 years ago the elder was twice as old as the younger. Find the age of each at present.

18. A father 40 years of age has 3 sons, the sum of whose ages is 20 years. In how many years will the sum of the sons' ages equal their father's age?

19. The age of a father is 3 years more than 5 times the age of his son, and the sum of their ages is 33 years. Find their ages.

20. There are four more girls than boys in a certain class, and three times the number of boys is greater by 9 than twice the number of girls. How many girls are there?

21. A parent divides \$2500 between 2 sons and 3 daughters, giving each son \$100 more than each daughter. Find the share of each.

22. A franc is worth 5 cents less than a mark; 5 francs and 7 marks are together worth \$2.63. Find the value of each coin in cents.

23. A rouble is worth 14 cents more than a guilder; 10 roubles and 5 guilders are together worth \$7.40. Find the value of each coin in cents.

24. A pound of tea is worth 5 cents more than 2 pounds of coffee; 4 lbs. of tea are worth 9 lbs. of coffee. Find the value of a pound of coffee.

25. A bushel of barley is worth 2 bushels of oats, and a bushel of wheat is worth 20 cents more than a bushel of barley; 2 bushels of wheat, 5 of barley and 10 of oats are worth \$10. Find the value of a bushel of oats.

26. A pound of tea is worth 6 cents more than 2 pounds of coffee; 4 lbs. of tea and 5 lbs. of coffee are together worth $2.45. Find the value of a pound of tea.

27. Divide $1.69 between *A*, *B* and *C*, giving *B* 15 cents less than twice as much as *C*, and *A* 1 cent more than *B* and *C* together.

28. *A*, *B*, *C* and *D* together have $7.70. *A* and *B* have together $4.70; *A* and *C*, $5.45; *A* and *D*, $3.95. How much has *A*?

29. A purse contains quarter-dollars and 10-cent pieces, 42 coins in all; the total value is $6.45. Find the number of quarters.

30. A purse contains a number of 10-cent pieces, as many francs and six more, each worth 19 cents. The francs are together worth $2.40 more than all the 10-cent pieces. How much are all the francs together worth?

31. A merchant bought 100 yards of cloth at $2.50 per yard. He sold part of it at $2.75 per yard and the remainder at $3 per yard, gaining on the whole $40. How many yards did he sell at $3 per yard?

32. A workman worked 40 days, part of the time at $1.60 per day and the remainder of the time at $1.80 per day. For the former period he received $13 more than for the latter period. How much did he receive in all?

33. A workman was engaged for 60 days on condition that he should receive $1.50 for each day he worked but should forfeit 75 cents for each day he was idle. At the end of the period he received $38.25. How many days was he idle?

34. A gentleman gave a number of children 10 cents each and had a dollar left. To have given them 15 cents

each he would have required a dollar more than he possessed. How much money had he?

35. A wine merchant has two kinds of wine, one worth 60 cents and the other 75 cents a quart. From these he wishes to make a mixture of 100 gallons worth \$2.75 a gallon. How many gallons of the former kind must he take?

36. Two casks contain equal quantities of water. From the first cask 40 quarts are drawn and from the second 35 gallons. One cask now contains twice as much as the other. How much did each cask at first contain?

37. A rectangle is 1 foot longer and 6 inches narrower than the side of a square. The area of the square is 48 sq. in. less than the area of the rectangle. Find the area of the square.

38. The sides of two squares differ by 3 inches and their areas differ by 117 square inches. Find the area of the smaller square.

39. The length of a field is twice its width; if 10 yards were added to its width and the same amount subtracted from its length, the area would be increased by 700 square yards. Find its original area.

EQUATIONS WITH FRACTIONS

110. Equations frequently occur with fractional coefficients. They may always be solved by the methods of the following examples.

Ex. 1. Solve the equation $\frac{x}{3} - 5 = 3\frac{1}{3} - \frac{x-1}{5}$.

Multiply each side by 15; this will not destroy the equality and will cause the fractions to disappear.

Therefore $\qquad 5x - 75 = 50 - 3(x-1)$

$$= 50 - 3x + 3.$$

Then $\qquad 8x = 128,$

and $\qquad x = 16.$

Since letters in algebra stand for numbers, the multiplication of fractions containing letters follows the ordinary rules of arithmetic.

Thus $\frac{x}{3} \times 15 = 5x$; $\frac{x-1}{5} \times 15 = 3(x-1)$, etc.

Observe carefully the negative sign before the fraction containing two terms in the numerator. When the denominator has been removed by multiplication the negative sign causes the sign of each term in the numerator to be changed as in the example given.

The multiplier which will cause all the fractions to disappear is evidently the least common multiple of the denominators. This process is known as "clearing an equation of fractions."

Ex. 2. Solve equation $\frac{1}{2}(x-3) - \frac{1}{3}(x-5) = 1 - \frac{1}{12}(x-8)$.

Multiply by 12, $\quad 6(x-3) - 4(x-5) = 12 - (x-8)$.

Remove brackets, $\quad 6x - 18 - 4x + 20 = 12 - x + 8,$

then $\qquad 3x = 18,$

and $\qquad x = 6.$

Observe the two different forms in which fractions containing literal expressions may be written, with the fractional part as a coefficient as in this example, or with the denominator written beneath the numerator as in Ex. 1. The meaning is the same in both cases. Such terms are multiplied by multiplying the fractional coefficient only. Art. 70.

The student should verify each solution.

Thus when $x=6$,

$$\frac{1}{2}(x-3)-\frac{1}{3}(x-5)=\frac{1}{2}(6-3)-\frac{1}{3}(6-5)=\frac{3}{2}-\frac{1}{3}=1\frac{1}{6}$$

$$1-\frac{1}{12}(x-8)=1-\frac{1}{12}(6-8)=1-\left(-\frac{1}{6}\right)=1\frac{1}{6},$$

which proves the solution correct.

EXERCISE XVIII

Solve and verify each result:

1. $2x+\frac{x-5}{3}=\frac{37}{3}.$

2. $\frac{x-1}{2}+\frac{x-2}{3}=5.$

3. $\frac{x-4}{4}=6-\frac{x}{3}.$

4. $\frac{x-1}{2}-\frac{x-2}{3}=\frac{2}{3}-\frac{x-3}{4}.$

5. $\frac{x+1}{2}+\frac{x+2}{3}+\frac{x-9}{4}=0.$

6. $\frac{x-5}{2}-\frac{x-4}{3}=\frac{6-2x}{3}.$

7. $\frac{5x+3}{3}-\frac{3x-7}{2}=10(5x-1)-\frac{19}{9}.$

8. $\frac{2x-1}{3}-\frac{5-4x}{4}=x+\frac{5}{6}.$

9. $\frac{2x-11}{7}+\frac{5x-3}{2}=x+\frac{1}{2}.$

10. $\frac{8x-15}{3}-\frac{11x-1}{7}=\frac{7x+2}{13}.$

11. $\frac{3x+5}{8}-\frac{21+x}{2}=5(x-3).$

12. $\frac{x}{4}-\frac{x-1}{3}=\frac{x-5}{12}+\frac{1}{2}.$

13. $\frac{2x-3}{12}+\frac{2-3x}{5}+\frac{3-4x}{8}=0.$

14. $\frac{2x+7}{3}+\frac{3-x}{8}=\frac{2(x+5)}{11}.$

15. $\dfrac{3x-5}{2}+\dfrac{4x-11}{3}-\dfrac{5x-37}{6}=0.$

16. $\dfrac{x}{4}+\dfrac{5}{3}(11-x)=\dfrac{1}{12}(34-11x).$

17. $\dfrac{3}{5}+\dfrac{2}{x}+\dfrac{7}{3x}-\dfrac{1}{5x}=\dfrac{8}{3}.$

18. $\dfrac{x+1}{2}-\dfrac{3}{x}=\dfrac{x}{3}-\dfrac{5-x}{6}.$

19. $\dfrac{3-x}{4}-\dfrac{2}{x}+\dfrac{x+6}{3}=\dfrac{x}{12}.$

20. $\dfrac{x}{2}+\dfrac{2x-5}{5}=\dfrac{9x^2-40}{10x}.$

21. $\dfrac{1}{3}(1-2x)-\dfrac{1}{4}(7-2x)+\dfrac{1}{6}(11-2x)+\dfrac{7}{12}=0.$

PROBLEMS

111. The solution of problems resulting in fractional equations is of the same general nature as the solutions already given.

Ex. 1. Bought a number of apples at 2 for a cent and twice as many more, lacking 5, at 5 for a cent. Sold them all at 5 for 2 cents, gaining 17 cents in all. How many did I buy?

Let $x =$ number at 2 for a cent,

then $2x-5 =$ " 5 "

and $\dfrac{x}{2} =$ number of cents for first lot,

" $\dfrac{2x-5}{5} =$ " " second lot.

Also $3x-5 =$ total number bought,

and $\dfrac{2}{5}(3x-5) =$ number of cents received.

Then $\dfrac{x}{2}+\dfrac{2x-5}{5}=\dfrac{2}{5}(3x-5)-17,$

solving $x = 60.$

Then $3x-5 = 175$, the total number bought.

Ex. 2. A saves 20% of his income; B has $1\frac{1}{2}$ times as large an income and saves 25% of it. A spends \$20 more in 6 mos. than B saves in a year. Find their incomes.

Let $x = A$'s income in dollars,

then $\frac{3x}{2} = B$'s " "

and $\frac{4x}{5} = A$'s expenses " per annum,

" $\frac{3x}{8} = B$'s savings per annum.

Therefore $\frac{2x}{5} - 20 = \frac{3x}{8}$,

from which $x = 800$, and $\frac{3x}{2} = 1200$.

Their incomes are, therefore, \$800 and \$1200 respectively.

Ex. 3. A rectangle is $\frac{2}{3}$ as wide as it is long, and its area is 25 sq. ft. less than the area of a square of equal perimeter. Find the area of the square.

Let $2x =$ the width of the rectangle in feet,

then $3x =$ " length " " "

and $10x =$ " perimeter of rectangle,

" $\frac{5x}{2} =$ side of square of equal perimeter,

" $6x^2 =$ area of rectangle in square feet,

" $\left(\frac{5x}{2}\right)^2 =$ " square "

Then $6x^2 + 25 = \left(\frac{5x}{2}\right)^2$,

" $24x^2 + 100 = 25x^2$,

" $x^2 = 100$, and $x = 10$.

Then area of square $= \left(\frac{5x}{2}\right)^2 = (25)^2 = 625$ sq. ft.

EXERCISE XIX

1. Find a number whose half exceeds its third by $2\frac{5}{6}$.

2. Two numbers differ by a unit, and if the larger be divided by 3 and the smaller by 5 the sum of the quotients will be 11. Find the larger number.

3. Two numbers differ by 9 and one of them is $\frac{5}{8}$ of the other. Find the larger number.

4. Divide 17 into two parts, such that a third of one part and a quarter of the other may together equal 5.

5. Find the number whose third part is as much less than 48 as its double is greater than 29.

6. It requires $6\frac{1}{4}$ minutes longer for a boy to walk to school at 3 miles per hour than at 4 miles per hour. How far is it to school?

7. Find a number such that when diminished by 7, one-third the remainder is greater by 3 than one-fifth of the original number.

8. A boy bought a number of marbles at 3 for a cent, and having lost 5 he sold the remainder at a half cent each, gaining on the whole 4 cents. How many did he buy?

9. A post stands with one-third of its length in the earth, one-half in the water and 6 feet above the surface of the water. Find its length.

10. A boy spent one-fifth of his money for candy, one-half the remainder for oranges and had 5 cents more than one-third of his original sum left. What had he at first?

11. The number of boys in a certain class is one more than one-half the whole, and the number of girls is 6 less than $\frac{5}{8}$. How many in the class?

12. Bought a number of apples at 2 for a cent and twice as many lacking one at 3 for a cent; the cost of the second lot was 8 cents more than the cost of the first lot. How many in all did I buy?

13. A journey of $26\frac{1}{2}$ miles was performed in $5\frac{1}{2}$ hours, part of it at 4 miles per hour and the remainder at 7 miles per hour. How many miles at each rate?

14. How far can a boy ride his wheel at 12 miles per hour and return on foot at 4 miles per hour to a point 2 miles from where he started, being $5\frac{1}{2}$ hours in all on the journey?

15. Divide $1000 into two parts, such that the interest on part of it at 5% and on the remainder at 6% may be $1.50 more than the interest on the whole at $5\frac{1}{4}$%.

16. Divide $780 into two parts, such that the simple interest on the first part for 3 months at 5% may be $1.50 less than the simple interest on the remainder for 5 months at 6%.

17. A purse contains dimes and quarters, 35 coins in all. The total value of all the quarters is $1\frac{2}{3}$ times the total value of the dimes. How many dimes in the purse?

18. One-third of a rouble is worth 1 cent less than a franc, and 4 francs are together worth 5 cents less than $1\frac{1}{2}$ roubles. Find the value of a rouble in cents.

19. A franc is worth 3 cents more than $\frac{2}{3}$ of a mark; a mark and a franc are together worth 43 cents. Find the value of each coin in cents.

20. Five years ago Ann was half as old as Mary; at present Mary is 3 times as old as she was when Ann was born. How old is Ann?

21. A rectangle is $\frac{3}{4}$ as wide as it is long and its area is $2\frac{1}{4}$ sq. ft. less than that of a square of equal perimeter. Find the area of the rectangle.

22. A rectangle is $\frac{5}{8}$ as wide as it is long, and if an inch is added to both length and width its area is increased by 91 square inches. Find its perimeter in feet.

23. Two workmen, A and B, have each the same income of which A saves 5% and B 10% annually. B spends \$15 more per month than A saves in a year. Find their income.

24. A teacher was able to save 15% of his salary, but his rent having been raised \$5 per month, his whole annual expenses are now 6 times what he formerly saved. Find his income.

25. A saves 20% of his income; B has $1\frac{1}{2}$ times as large an income and saves 25% of it. A spends \$20 more in 6 months than B spends in 4 months. How much does A save in a year?

26. A workman worked 64 days for \$148, part of the time receiving \$2 per day and for the remainder \$2.50 per day. How much money did he earn at \$2 per day?

27. A workman saves 10% of his income. His pay is increased 10% and his expenses rise $12\frac{1}{2}$%; his weekly savings are now 15 cents less than before. Find his original weekly wage.

28. A teacher gets an increase of \$50 per annum for two years. The first year he saves $\frac{1}{4}$ of his salary, the second year he saves a third, and the last year he saves one-half. His whole saving is equal to his salary for a year and two months at the original rate. Find his first year's salary.

CHAPTER V

SIMPLE EQUATIONS: TWO UNKNOWN QUANTITIES

(ELEMENTARY)

112. In the solution of problems in which two unknown numbers are to be found, we may use two letters, one to represent each number. We must then obtain from the problem two distinct statements, each of which will furnish an equation. From these equations the unknown numbers are to be found.

Ex. Find the two numbers whose sum is 10 and difference 2.

Let	$x =$ the larger number,	
and	$y =$ " smaller "	
Then	$x + y = 10$	(1)
and	$x - y = 2.$	(2)
Adding	$2x = 12,$	(Ax. 1.)
or	$x = 6.$	
In (1) for x write 6,	$6 + y = 10,$	
or	$y = 4.$	

Then 6 and 4 are the numbers required.

Observe carefully the two facts given in this problem, (1) the sum is 10, (2) the difference is 2, and note how each equation expresses one fact in algebraical symbols.

113. In the preceding example, if but one statement had been made regarding the two numbers no definite result could have been obtained.

For if $x + y = 10$ alone be given,

then $x = 7,\ x = 6,\ x = 12$, etc.,

$y = 3,\ y = 4,\ y = -2$,

all satisfy the required condition, since the sum of each pair is 10.

Similarly if $x - y = 2$ alone is given,

we have $x = 8,\ x = 6,\ x = 1$, etc.,

$y = 6,\ y = 4,\ y = -1$,

each pair of which satisfy the given equation. But when both equations are to be satisfied *at the same time, i.e.*, by the same pair of numbers, there is but one pair, 6 and 4, which can be chosen.

114. Independent equations. Two equations which express different facts, *i.e.*, two facts, one of which cannot be inferred from the other, are said to be **independent.** The two equations of Art. 110 are independent, since from the fact that the sum of two numbers is 10 we cannot infer that their difference is 2. But the two equations

$$x + y = 10$$
$$2x + 2y = 20$$

are not independent: the second equation is a mere repetition of the first.

115. Simultaneous equations. Two independent equations which are to be satisfied by the same values of two unknown quantities are called **simultaneous equations.**

The two equations of Art. 112 are simultaneous equations.

116. Elimination. From two simultaneous equations containing two unknown quantities it is generally possible to obtain a new equation containing but one unknown quantity. The quantity which does not appear in the new equation is said to be **eliminated,** and the process by which the new equation is obtained is called **elimination.**

By adding the two equations of Art. 112 the y was eliminated and the new equation contained x alone. Its value was then easily found.

117. Solution of simultaneous equations. The solution of simultaneous equations is effected by eliminating one of the unknown quantities and solving the resulting equation by the methods already given.

Ex. 1. Solve the equation $2x + 3y = 21,$ (1)

$$5x + 2y = 25. \qquad (2)$$

Multiplying the first equation by 2 and the second by 3 we get

$$4x + 6y = 42, \qquad (3)$$

$$15x + 6y = 75. \qquad (4)$$

Subtracting (3) from (4), $11x = 33,$

or $x = 3.$

Similarly multiplying (1) by 5 and (2) by 2

we get $10x + 15y = 105,$ (5)

$$10x + 4y = 50. \qquad (6)$$

Subtracting (6) from (5), $11y = 55,$

or $y = 5.$

Having found the value of x, we might have substituted its value, 3, in either of the given equations, and then y

would have been easily found. Thus for x write 3 in equation (1),

we have $$6 + 3y = 21,$$
from which $$y = 5 \text{ as before.}$$

This process, known as **substitution**, is usually the most satisfactory method.

Ex 2. Solve $$\frac{x+y}{7} - \frac{2y-x}{3} = 3, \qquad (1)$$
$$\frac{3y+2x}{4} + \frac{9(x-1)}{8} = \frac{x}{2}. \qquad (2)$$

Multiply (1) by 21, $$3x + 3y - 14y + 7x = 63,$$
collecting terms, $$10x - 11y = 63. \qquad (3)$$
Multiply (2) by 8, $$6y + 4x + 9x - 9 = 4x,$$
collecting terms, $$9x + 6y = 9,$$
or $$3x + 2y = 3. \qquad (4)$$
Multiplying (3) by (2) and (4) by 11,
$$20x - 22y = 126, \qquad (5)$$
$$33x + 22y = 33. \qquad (6)$$
Adding (5) and (6), $$53x = 159,$$
or $$x = 3.$$
Substituting 3 for x in (4), $$9 + 2y = 3,$$
or $$y = -3.$$

VERIFICATION

$$\frac{x+y}{7} - \frac{2y-x}{3} = \frac{3-3}{7} - \frac{-6-3}{3} = 0 - (-3) = 3,$$

and $$\frac{3y+2x}{4} + \frac{9(x-1)}{8} = \frac{-9+6}{4} + \frac{18}{8} = -\frac{3}{4} + \frac{9}{4} = \frac{3}{2} = \frac{x}{2},$$

which proves the values found for x and y to be correct.

118. The ability to solve simultaneous equations quickly and correctly is obtained only by observation and experience, but the following general directions may be of service to the learner :

1. Clear each equation of fractions, remove brackets, collect the terms and strike out any factor which may be common to all the terms in either equation.

2. To eliminate a letter, find the L. C. M. of its coefficients in the two equations and multiply each equation by the quotient obtained by dividing the L. C. M. by the coefficient of that letter.

3. Subtract or add the resulting equations according as the signs of the coefficients of the letter to be eliminated are alike or different.

4. Substitute the value of the letter thus found for that letter in one of the preceding equations, choosing the one in the simplest form and with the smallest coefficients and thus find the value of the remaining letter.

EXERCISE XX

Solve the equations and verify the results obtained.

1. $x+y=20,$
   $x-y=4.$

2. $x+y=25,$
   $x+2y=28.$

3. $2x+y=35,$
   $x+2y=37.$

4. $3x-y=16,$
   $2x+5y=141.$

5. $5x-2y=25,$
   $2x+3y=-9.$

6. $8x+3y=37,$
   $12x+5y=59.$

7. $2(x-1)+y=12,$
   $4x+3(y-1)=29.$

8. $3(x-5)+6=5y,$
   $4(y+1)+8=3x.$

9. $2(x+3)+7=y+23,$
$3(x-y)-5=4x-3.$

10. $x-y=7(x+y),$
$5x+7y=1.$

11. $\frac{x}{2}+\frac{y-1}{3}=6,$
$\frac{x-y}{4}+\frac{x}{3}=1.$

12. $\frac{x-y}{3}+1=\frac{x+y}{5}-\frac{7}{3},$
$\frac{x}{3}-\frac{y}{2}=0.$

13. $\frac{2x}{3}+\frac{y-1}{2}=\frac{x+y}{2},$
$\frac{x-1}{2}-\frac{y}{5}=\frac{y-x}{2}-1.$

14. $2(x-1)+\frac{2y}{5}=21,$
$3(y+1)-\frac{x}{11}=x-\frac{3}{2}.$

15. $\frac{x-1}{3}+\frac{y+1}{5}=\frac{x+y}{3},$
$x+5y=0.$

16. $\frac{y-3}{7}+\frac{2y-3}{5}=\frac{y}{2},$
$2(x-y+5)=3(y-x).$

17. $\frac{x+2y}{5}-\frac{y}{10}=\frac{2x+3y}{15},$
$\frac{x+5}{2}-1=0.$

18. $12(x-y)=\frac{3x+4y}{2},$
$\frac{x+1}{2}=4y-x.$

19. $\frac{1}{3}(x+7)+\frac{1}{5}(y+2)=3\frac{1}{2},$
$\frac{1}{5}(x+y)-\frac{1}{3}(x-y)=0.$

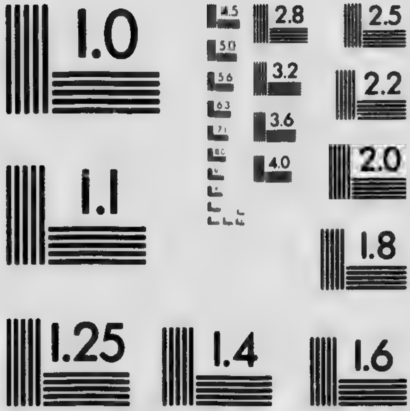
20. $\frac{1}{7}(3x+2y)=x-10,$
$\frac{1}{9}(3x-2y)=y+\frac{9x}{16}.$

21. $\frac{1}{3}(5x+7y+2)-\frac{1}{4}(3x+4y+7)=x,$
$\frac{1}{4}(7x+3y+4)-\frac{1}{5}(6x+5y+7)=y.$

22. $\frac{1}{3}(2x-3y-1)+\frac{1}{5}(x+y-2)=2x-\frac{y}{5},$
$\frac{1}{2}(5x+2y+1)-\frac{1}{3}(3-x-y)=\frac{1}{6}(y-2x).$

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



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PROBLEMS

119. The following are additional examples of the use of two letters in the solution of problems. In many cases the solution may also be effected by the skilful use of a single letter. The learner is recommended to occasionally solve the same problem by each method and to compare the various steps in the two solutions.

Ex. 1. A bill of \$5.55 was paid in quarter-dollars and 10-cent pieces, 27 coins in all. How many of each were used?

Let $x =$ the number of quarter-dollars,
and $y =$ " 10-cent pieces.
Then $25x =$ the value of the quarters in cents,
and $10y =$ " 10-cent pieces.
Then $x + y = 27$, the number of coins,
and $25x + 10y = 555$, their value in cents.

Solving these equations in the usual way, $x = 19$, and $y = 8$, the numbers required.

Ex. 2. A number consisting of two digits is equal to 5 times the sum of its digits, and if 9 be added, the resulting number consists of the same digits interchanged. Find the numbers.

Let $x =$ the tens' digit and $y =$ the units' digit.
Then $x + y =$ the sum of the digits,
$10x + y =$ the original number,
and $10y + x =$ the second "

The equations are $10x + y = 5(x + y)$,
and $10x + y + 9 = 10y + x$.
Solving, we get $x = 4$ and $y = 5$.
The required number is, therefore, 45.

Ex. 3. A saves $\frac{1}{5}$ of his daily pay and B saves $\frac{1}{4}$ of his; together they save \$1.25 per day. A receives an increase of 10% and B an increase of $12\frac{1}{2}$%, and now A saves 4 cents per day more than B. Find the daily wage of each.

Let x = number of cents received per day by A,
and y = " " " " " " " B.

Then
$$\frac{x}{5}+\frac{y}{4}=125. \quad (1)$$
and
$$\frac{1}{5}\left(\frac{11x}{10}\right)-\frac{1}{4}\left(\frac{9y}{8}\right)=4. \quad (2)$$
Simplifying
$$4x+5y=2500 \quad (3)$$
$$176x-225y=3200. \quad (4)$$
Multiplying (3) by 45, $180x+225y=112500.$

Adding (3) and (4), $356x=115700,$

or $x=325,$

then $y=240.$

Their incomes are, therefore, \$3.25 and \$2.40 respectively.

EXERCISE XXI

1. The sum of two numbers is 47, and 3 times the smaller number is greater by a unit than twice the larger. Find the numbers.

2. The sum of two numbers is 5 times their difference, and the double of the smaller is greater than the larger by 4. Find the numbers.

3. Two pounds of tea and 3 pounds of coffee are together worth \$1.43, while 3 pounds of tea and 2 pounds of coffee are worth \$1.62. Find the value of a pound of each.

4. Two men together earn 90 cents more per day than 3 boys, and 4 men together earn 60 cents more per day than 7 boys. Find the daily wage of a man and a boy.

5. Two apples cost one-half as much as 3 oranges; a dozen apples and 21 oranges together cost a dollar. Find the price per dozen of apples and oranges.

6. Two bushels of oats weigh 8 lbs. more than one bushel of wheat; 2 bushels of wheat weigh 18 lbs. more than 3 bushels of oats. By how much does the weight of 3 bushels of wheat exceed the weight of 5 bushels of oats?

7. A and B play for a stake of $10, to be furnished by the loser. If A wins he will then have twice as much as B, but if B wins he will have 3 times as much as A. How much money had each at first?

8. Tom saves 25% of his week's wages and Dick saves 20% of his; together they save $6 per week. Tom's expenses are $\frac{3}{4}$ of Dick's expenses. Find their weekly wage.

9. A saves half his income, B saves one-third of his income; together they save $1.80 per day. If their incomes were interchanged and each saved the same fraction of his income as before, A would save 10 cents a day more than B. How much does each save per day?

10. Paid a dollar for some apples at 3 cents each and some oranges at 5 cents each. Sold two-fifths of the apples and one-fourth of the oranges at cost for 34 cents. How many of each did I buy?

11. A and B together earn $5.75 per day. If A's wages were reduced 20% and B's raised 20%, A would still have 10 cents a day more than B. How much does each earn per day?

12. The sum of the ages of A and B is $\frac{2}{3}$ of the sum of the ages of C and D. Two years ago the sum of the ages of A and B was one-half the sum of the ages of C and D. Find the sum of the ages of all four at present.

13. A farmer sold wheat at a dollar a bushel and barley at 80 cents, the average price for the whole being 88 cents. The total value of the barley was \$10 more than the total value of the wheat. How many bushels did he sell in all?

14. A purse contains quarters and half-dollars, \$13.75 in all. The total value of the half-dollars is greater by \$5.25 than the total value of the quarters. How many coins are in the purse?

15. A number consisting of two digits is 4 times the sum of its digits. If 27 be added the resulting number will consist of the original digits interchanged. Find the number.

16. A number consisting of two digits is greater by 2 than 3 times the sum of its digits. The sum of its digits is double their difference. Find the number.

17. Prove that the sum of any two numbers consisting of the same pair of digits interchanged is divisible by 11, and the difference of the numbers is divisible by 9. State in words the quotient in each case.

18. Show that a number consisting of two digits, whose sum is three times their difference, is equal to either 4 times the sum of its digits or to 7 times their sum. Distinguish between the two cases.

19. A certain fraction becomes equal to $\frac{1}{2}$ when a unit is added to the numerator, but equal to $\frac{1}{3}$ if a unit be added to the denominator. Find the fraction.

20. The value of a certain fraction becomes $\frac{1}{4}$ when a unit is added to its numerator and to $\frac{1}{8}$ if 8 be added to its denominator. Find the fraction.

21. A bill is exactly paid by 8 marks and 12 guilders or by 13 marks and 9 guilders. How many marks would pay the bill?

22. A freight car carries 13 bales of cotton and 33 casks of wine as a full load. When 9 casks and 5 bales have been removed the car is still two-thirds filled. How many bales would fill the car?

23. The area of a rectangle will be unchanged if its width be increased 3 ft. and its length diminished 4 ft., but if its width be diminished by 3 ft. and its length increased 5 ft., the area will be reduced by 15 sq. ft. Find its original length and breadth.

24. A boy can ride a bicycle $1\frac{1}{2}$ miles further in three hours than he can walk in 7 hours. He can ride from home to school in 15 minutes and return on foot in $36\frac{2}{3}$ minutes. How far is it to school?

25. A journey was performed in $4\frac{1}{2}$ hours, a part of it at 4 miles per hour and the remainder at 10 miles per hour. If the distances travelled at the two rates were interchanged, the time required would be 27 minutes greater. Find the whole distance travelled.

EXAMINATION PAPERS

I

1. Find the value of $(x-a)^2+(x-b)^2-2(x-a)(x-b)$ when $x=5$, $a=4$, $b=0$.

2. Draw a rectangle whose length is x feet and width y feet. Write in three different ways its perimeter in feet. Write its area in square feet and in square inches.

3. From the sum of $2a-3b+c$, $2(b-c)-a$ and $3b-2\left(c+\frac{1}{2}a\right)$ take the sum of $2a-3(b-c)$ and $c+2(a-b)$.

4. Multiply $(a+b)^2$ by $(a-b)^2$, add $a^2b^2-b^4$ to the product and divide the final result by $a-b$.

5. Divide $a^2(a-b)-b^2(a-b)+ab(a+b)$ by $a+b$.

6. Solve the equation

$$3x-[5-\{2x+3(1-x)+2\}-3]=10.$$

7. A fish was caught whose tail weighed 5 lbs.; its head weighed as much as its tail and $\frac{1}{3}$ of its body, and its body weighed as much as its head and tail. Find weight of the fish.

II

1. Find the value of $\dfrac{a^2+bx^2}{ax-b-c^2}$ when $a=3$, $b=5$, $c=2$, $x=1$.

2. If $2^x=16$, find the values of $3x$, x^3 and 3^x.

3. Add $a^2-2ab-\dfrac{10b^2}{15}$, $2b^2-3a\left(a+\dfrac{1}{3}b\right)$, $\dfrac{2b}{4}(a-2b)+2a^2$.

4. Simplify

$$(2x+5)(x-3)+(1-2x)(3x+1)-3(2x-1)(2-x).$$

5. Divide

$$(a-2b)(a-3b)+2b(a-38b) \text{ by } 2(a+3b)-(a-b).$$

6. Solve the equation $\dfrac{1}{3}\left(\dfrac{3x}{4}+6\right)-\dfrac{4\frac{1}{2}-x}{3}=\dfrac{x}{2}\left(\dfrac{4}{x}+\dfrac{1}{6}\right)$.

7. Tom saves \$10 per week and Dick saves \$12 per week; Harry saves \$15 per week but starts two weeks after the other two. When will Harry's savings be as much greater than Tom's as they are less than Dick's savings, and how much money will each then have?

III

1. Find the value of $\sqrt{1+\dfrac{x-1}{1-y}}-\sqrt{\dfrac{1}{6}\left(\dfrac{1}{1-x}+\dfrac{1}{y-1}\right)}$ when $x=\dfrac{1}{3}$, $y=\dfrac{1}{4}$.

2. A rectangle is 6 feet longer than it is wide, and a square has the same perimeter as the rectangle. How much greater is the area of the square than that of the rectangle?

3. Express in words the result when the difference of any two number (1) added to their sum, (2) subtracted from their sum.

4. Simplify $\frac{1}{2}\left(\frac{2x}{3}+4\right)-\frac{1}{3}\left(7\frac{1}{2}-x\right)-\frac{x}{2}\left(\frac{6}{x}-1\right)$ and find the value of the result when $x=3$, and when $x=5$.

5. Divide

$$(a-x)(a-2x)+(x-a)(x- \quad)-2(a-x)^2 \text{ by } x-a.$$

6. Solve equation $\frac{x}{2}(2-x)-\frac{x}{4}(3-2x)=\frac{x+10}{6}$ and verify the result.

7. A and B play marbles. At the end of the first game A had twice as many as he had at first. At the end of the second game B had twice as many as he had at the close of the first game, and then each had the same number. If they together had 80, how many had each at first?

IV

1. Find the value of $a^2b-\frac{a}{b^2}-\left(\frac{a}{b}\right)\left\{\frac{1}{a^2}-\frac{1}{ab}+\frac{1}{b^2}\right\}$ when $a=-1$, $b=2$.

2. A rectangle is $2x+3$ feet long and width 8 feet less; find its perimeter, its area and the area of a square of equal perimeter.

3. Simplify $a(a+b-1)-b(a-b+2)-(a-b)(a+b-3)$.

4. Multiply $(a-b)^2+(b-c)^2+(c-a)^2$ by $a+b+c$.

5. Divide $(a^2+b^2)^2-a^2b^2$ by $(a+b)^2-ab$.

6. Solve equation $\frac{1}{2}\left\{\frac{1}{2}\left(\frac{x}{2}-\frac{1}{2}\right)-\frac{1}{2}\right\}-\frac{1}{2}=1$.

7. At a baseball game Tom made 5 more runs than he was years old; Dick, who was two years older, made twice as many runs as he was years old, beating Tom by 9 runs. How old was Tom?

V

1. Find the value of $ab(a-b)+bc(b-c)+ca(c-a)$ when $a=2$, $b=-3$, $c=5$.

2. If $x=a+b$, $y=a-b$, find the value of x^2-xy+y^2.

3. Divide

$$(x-1)^3+(x-2)^3+(x-3)^3-3(x-1)(x-2)(x-3) \text{ by } 9(2-x).$$

4. Simplify

$$(a-b)(a-b-x)(a+2b-2x)+b(b-x)(3a-2b-2x)$$

when for x we write a.

5. If $\frac{1}{2}(x-1)+\frac{1}{3}(x-2)=\frac{1}{4}(x-3)$, find the value of

$$\frac{1}{2}(x+1)+\frac{1}{3}(x-2).$$

6. Solve equations $x-\frac{1}{7}(y-2)=5$, $4y-\frac{1}{3}(x+10)=3$.

7. A boy spent 50 cents in buying apples at the rate of 3 for 5 cents and oranges at 40 cents a dozen. Had the numbers of apples and oranges been interchanged the cost would have been 20 cents more. How many of each did he buy?

VI

1. If $\frac{2}{3}(6x-5)=\frac{x}{6}-\frac{1}{2}(2-3x)$, find the value of

$$\frac{1}{3}(1-2x)+\frac{1}{4}(x-1)^2.$$

2. Find the algebraical expression which when divided

by $a^2 - ab + b^2$ gives $a^2 + ab + b^2$ for quotient with $-b^4$ as remainder.

3. Simplify

$$(1-a)(1-b)(1-c) + a(1-b)(1-c) + b(1-c) + c.$$

4. Divide $(a+b)(b+c)(c+a) + abc$ by $a+b+c$.

5. Solve equation $\frac{1}{5}(2x+1)^2 - \frac{1}{20}(4x-1)^2 = \frac{15}{8} + \frac{3(4x+1)}{40}$.

6. Solve equations $\frac{6x-2y}{5} - \frac{y-x}{10} = \frac{2x-1}{10}$

$$y - \frac{1}{3} + \frac{x}{4} = 10 - \frac{y-2x}{3}.$$

7. Five francs are together worth one cent more than 2 florins, and if the value of a franc were decreased one cent, 3 francs would be worth 7 cents more than a florin. Find the value of each coin in cents.

VII

1. If $x = \frac{1}{3}$, $y = -\frac{1}{2}$ and $\frac{2}{3x} - \frac{3}{2y} + \frac{5}{z} = 6$, find the value of z.

2. Simplify $(x+1)\{x(x-1)+1\} - (x-1)\{1+x(1+x)\}$.

3. Multiply $(a+b)^2 + (b+c)^2 - (a-c)^2$ by $a-b+c$.

4. Divide $(1+b)(1-b) + a + ab(a+b)$ by $1+a+b$.

5. Solve equation $a(x-a^2) + b(x-b^2) = 0$ and verify the result.

6. The perimeter of a room is 44 feet. If it were 2 feet longer and 1 foot wider the area of the floor would be 34 sq. ft. greater. Find its length and width.

7. Two-thirds of a shilling is worth 3 cents less than a franc, and a half a franc is worth half a cent more than $\frac{3}{8}$ of a shilling. Find the value of each coin in cents.

VIII

1. Simplify

$$(a+b)\{2a-3(a-2b)-b\}-(a-b)\{2(b-3a)+2a-3b\}.$$

2. Multiply $x^2-(a+b)x+ab$ by $x^2+(a-b)x-ab$ and divide the result by $(x-b)^2$.

3. Divide

$$(x-a)^2+(y-b)^2-(ay-bx)^2+(a^2+b^2-1)(x^2+y^2-1)$$

by $ax+by-1$.

4. If $x=a+2b-3c$, $y=a-b+3c$, find the value of $x^2+xy-2y^2$ in terms of a, b and c.

5. Solve the equation $\frac{2x}{3}\left(1-\frac{5}{x}\right)+\frac{3x}{4}\left(1-\frac{4}{x}\right)=\frac{5}{4}(x-4)$, and verify the result.

6. A number consisting of two digits is greater by 2 than 5 times the units' digit. If the digits be reversed the resulting number will be greater by 3 than 7 times the sum of its digits. Find the number.

7. *A* loses $\frac{1}{4}$ of his money and *B* gains an amount equal to $\frac{3}{8}$ of what *A* had at first, and now they have equal sums. If *A* should now give *B* \$50 he would have only one-half as much left as *B* would then have. How much had each at first?

ANSWERS AND RESULTS

Exercise I. Page 15

1. 8, 16, 12, 64, 48, 144.
2. 15, 45, 75, 225, 34, ...
3. 6, 5, 7, 12, 15, 20.
4. 7, 5, 9, 5, 6.
5. 6.
6. 8.
7. 11.
8. 20.
9. 9.
10. 90.
11. 26.
12. 16.
13. 25.
14. 88.
15. 5.
16. 0.
17. $\frac{1}{8}$.
18. 10.
19. 8.
20. $1\frac{3}{8}$.
21. 16.
22. $7\frac{1}{2}$.

Exercise II. Page 17

2. 3, 5, 7, 8.
3. 7, 11, 6, 9, 12.
4. 5, 4, 6, 5.
6. 4^3, 2^6.
7. 3^4, 9^2; 3^6, 9^3.
8. 5^4, 4^4.
9. 32, 25, 16, 27, 1, 15625.
10. 8, 4.
11. 32, 8, 9, 3.
12. 4, 3.
13. 4, 3, 27, 4.
14. 36.
15. 64.
16. 248, 7.
17. 6, 30.
18. 12.

Exercise III. Page 19

1. $x+5$, $5x$.
2. $10-7$, $10-x$.
3. $n+2$, $n+x$.
4. $n+5$, $n+y$.
5. $2x+2$.
6. $2a+2b$, $a-b$.
7. $m-p-q$, $m-(p+q)$.
8. $3n$, $3nx$.
9. $2n+3p$.
10. $100x$, $\frac{x}{100}$.
11. $100b-nx$, $b-\frac{nx}{100}$.
12. $12x+y$, $36x+12y$.
13. $5m$, mx, $\frac{mp}{60}$.
14. $\frac{m}{4}$ hrs., $\frac{m}{x}$ hrs.
15. $nq+mp$, $\frac{(nq+mp)x}{100}$.
16. $\frac{x+y}{6}$, xy, $\frac{xy}{144}$.
17. $4x+2$, $\frac{2x+1}{6}$.
18. $x+y+10$, $x-y$.

19. $7(10)+5$; $10x+y$. **20.** 4, 11, 5.
21. $35x+50$. **22.** 10. **23.** $6x^2$, 3.
24. xyz, $2(xy+yz+xz)$, $4(x+y+z)$. **25.** $2n+1$, $2n-1$.
26. $xy+r$. **27.** $(a+b)^2=a^2+b^2+2ab$.
28. $(a-b^2=a^2+b^2-2ab$. **29.** $\dfrac{a^3-b^3}{a-b}=a^2+b^2+ab$.
30. $(a+b)(a-b)=a^2-b^2$.

Exercise IV. Page 24

1. $+18$, -90. **2.** 30 ft. north, 15 ft. south, $7\frac{1}{2}$ ft. south, 1 ft. north. **3.** 5 ft. up, 10 ft. down.
4. \$2.50 gain ; \$3.25 loss. **5.** $-2\frac{3}{4}$, $+3\frac{2}{5}$, $+4\frac{1}{8}$, $-2\frac{1}{2}$.
6. -1, $+2$. **7.** $-2a$, $3a$. **8.** -10, $+3$, -4.
9. \$12 cash $=+12$, \$5 debt $=-5$. **10.** $+17$, $-30\frac{1}{2}$.
11. -2 oz., 14 oz., 2 oz. **12.** Halved, doubled, sign changed.

Exercise V. Page 30

1. $+8$, -8, -2, $+2$, -3, $+3$.
2. $+50$, -50, $+18$, -18, $+10$, -10.
3. a, $9b$, $-4a^2$, $-10ab$, $24xy$, $7m$. **4.** $4x^2$.
5. $-7x^2+3y^2$. **6.** $-6m^2-10n^2$. **7.** $-3ab$.
8. $-a^2b-4ab^2$. **9.** $-6(a+b)$. **10.** 6, -8, -12.
11. $10-25+5=-10=10$ miles west, the end of the journey; $10+25+5=40$ miles, the distance travelled.

Exercise VI. Page 32

1. $6a+4b-4c$. **2.** $2a+4b+11c$. **3.** $9a-5b+2c$.
4. $8a-2b-4c-4x$. **5.** $5ab+2ac+2bc$. **6.** $-ax-3bx+cx$.
7. 0. **8.** $7(a+b)$. **9.** $a+b-c$. **10.** $9(a^2+b)$.
11. $6a(b+c)+e+x$. **12.** $9a-4b-17c-12d+12e$.
13. $3(a^2+b^2)+13ab$. **14.** $-a^3+12a^2b-2ab^2$.
15. $2a^4-a^3b-5a^2b^2+5ab^3-7b^4$. **16.** $15x-7y+3z$.
17. $-4x-13y+23z$. **18.** $-4a+6b+6c$.
19. $-11a+13b+10c$.

Exercise VII. Page 36

1. $8, \ -8, \ -2, \ +2, \ 3, \ -3$. 2. $-9, 14, 3, -7, -4, +7$.
3. $10x, \ +y, \ 19a^2, \ -9ab, \ 3a-5b, \ 4x+3y$.
4. $4ax, \ 7by, \ 4a$. 5. $-2x^2, \ -10y^2, \ 2xy$. 6. $-10a^2, \ 0$.
7. $-9a+5b$. 8. $5, \ -9, \ -15$. 9. $-2, 11, -9, -9$.
10. $-10(x-y)^2$. 11. $+44-(-23)=67°$.

Exercise VIII. Page 37

1. $2a-4b+4c$. 2. $-a+4b+d$. 3. $-2b+3c-x$.
4. $4x^3-7x^2y+3xy^2$. 5. $2+3x-6x^3$.
6. $2a^2-2b^2+2bc-2ac$. 7. $3x^2-4xy+8y^2+2xz-yz$.
8. $2x^2+xy$. 9. $-x^2y-4y^3$.
10. $-a^2-b^2+c^2-ab+2bc+2ac$. 11. $-a+2b-c+ab^2$.
12. $-x^3+15x^2y+7y^3$. 13. $3(a-b)+13(x-y)$.
14. $2(a+b)+(c+d)+2(x-y)+p+q$. 15. $11a-4b-3c$.
16. $a-4b-2c-d-6e$. 17. $a-16c-5d+21e$.
18. $3a-3b-c$.

Exercise IX. Page 40

1. $2a$. 2. $2b$. 3. $2a-2c$.
4. $-b+2c$. 5. $6a-3c$. 6. $3a-4b+4c$.
7. $2a-4b-c$. 8. $x+2a+3$. 9. $x-4a$.
10. $-2y-2$. 11. $2a-b$. 12. $x-y+3z$.
13. $-2b+2c$. 14. $7a-6b$. 15. $5a-4b$.
16. $(a-b)+(c+d)-(e+f)$; $(a-b+c)+(d-e-f)$.
17. $a-(b-c-d)-(e+f)$; $a+(-b+c+d)+(-e-f)$.
18. $a-\{(b-c)-(d-e)+f\}$. 20. 0.

Exercise X. Page 46

1. $-15, \ -15, \ 15, \ -77, \ 40$.
2. $-15x^2, \ -6a^2bc, \ -20x^2y, \ 6a^3b^3, \ 5a^3bx$.
3. $-x^3y^2, \ 91x^2y^3z^2$. $x^3y^4, \ -8m^2n^2x, \ -35a^3b^3c$.
4. $-30a^4b^4c^4$. 5. $120a^3b^3x^3$. 6. $25, 8, 54$. 7. $0, 32$.
8. $-1, 5$. 9. $9a^6, \ 8a^6, \ 16a^4b^8, \ -243a^{10}b^{15}$.
10. $-3, 7, -10, -70, 29$. 11. 120.

Exercise XI. Page 40

1. $2x^3 - 4x^2 + 6x$.
2. $-9x^3 - 12x^2 + 6x$.
3. $2x^2y - 4xy^2 + 2y^3$.
4. $-8a^3b + 12a^2b^2 - 4ab^3$.
5. $-x + 2x^2 - 3x^3$.
6. $x^2y^2z + xy^2z^2 - x^2yz^2$.
7. $-21a^3bx^2 + 6a^2b^2xy + 9abx^2y^2 - 12ab^3xy$.
8. $-6a^3bc + 9ab^3c + 3abc^3 + 6a^2bc^2 + 12ab^2c^2 - 6a^2b^2c$.
9. $a^2b - a^3b + a^2b^2 - a^3bc + a^2b^2c - a^3b^2c$.
10. $8x^3 - 11x^2 + 12x$.
11. $-3x^3 - 16x^2 + 11x - 4$.
12. $6ab - 13b^2$.
13. $2a^3 - 11a^2b + 10ab^2 - 4b^3$.
14. $14a - 4b - 14c$.
15. $-2bc + c^2$.
16. $2x^2 + 10xy - 12y^2$.
17. $2px$.
18. $2bx + 2by$.
19. 0.
20. $cx + ay + bz$.

Exercise XII. Page 52

1. $6x^3 - 7x^2 + 11x - 6$.
2. $2x^3 + 3x^2 - 8x + 3$.
3. $20x^3 - 18x^2 - 25x - 6$.
4. $-3x^3 + 5x^2 + 7x - 10$.
5. $x^3 - 8$.
6. $a^3 + 1$.
7. $a^3 - b^3$.
8. $a^3 + b^3$.
9. $a^4 - 4a^2 + 12a - 9$.
10. $4a^4 - 13a^2b^2 + 9b^4$.
11. $6x^5 - x^4 + 4x^3 + 2x^2 - 7x + 2$.
12. $6x^5 - 5x^4 - 32x^3 + 5x^2 + 19x - 5$.
13. $1 - 4x^2 + 12x^3 - 5x^4 + 8x^5 - 12x^6$.
14. $2 - 8x + 13x^2 - 11x^3 + 6x^4 - 3x^5 + x^6$.
15. $-3x^5 + 8x^4 + x^3 + 3x^2 + 10x - 3$.
16. $x^6 - 1$.
17. $a^8 + 2a^6 + 3a^4 + 2a^2 + 1$.
18. $x^8 - 3x^7 - 2x^6 + 11x^5 - 14x^4 + 3x^3 + 16x^2 - 29x + 21$.
19. $x^3 + 3xy - y^3 + 1$.
20. $a^3 + b^3 + c^3 - 3abc$.
21. $2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4$.
22. $x^3 - 8y^3 + z^3 + 6xyz$.
23. $a^8 - 4a^6b^2 + 6a^4b^4 - 4a^2b^6 + b^8$.
24. $x^{12} - x^9y^3 + x^6y^6 - x^3y^9 + y^{12}$.
25. $x^2 + 19x + 4$.
26. $5 + x$.
27. -12.
28. $2a^2 - 2b^2$.
29. 0.
30. $a^3 - b^3$.
31. $b^4 + 2a^2b^2$.
32. $2b^4 - 32$.

Answers and Results

Exercise XIII. Page 56

1. $-4,\ -4,\ 4,\ 4.$
2. $-54,\ -30,\ 5,\ -21.$
3. $4ab^2,\ -4a^2b,\ 7x,\ -6n^4.$
4. $-21x^4y^5,\ -4ax^2,\ -5b^3,\ -6a^3b^2c^2.$
5. $3ab,\ ab.$
6. $ab,\ -15yz^2.$
7. $2x^2 - 3x + 4.$
8. $-3y^2 - 4y + 2.$
9. $a^2 - 2ab + 3b^2.$
10. $-a^2 + 2ab + 4c.$
11. $3x^3 - 5x^2y + 6xy^2 + y^3.$
12. $7x^2z^2 - 9xy + 8y^2z.$
13. $3 - 4(a+b) + 5(a+b)^2.$
14. $x^2 - z(x-y) + y^2.$
15. $(3b-a)(a+b).$

Exercise XIV. Page 60

1. $x + 7.$
2. $x - 3.$
3. $x - 8.$
4. $x - 9.$
5. $2x + 3.$
6. $x^2 + 2x - 3.$
7. $a + b.$
8. $a^2 + ab + b^2.$
9. $a^2 - ab + b^2.$
10. $a - b.$
11. $x^2 + 3x + 2.$
12. $2x^2 - 3x + 7.$
13. $x + 5.$
14. $a^3 - 3a^2 + 3a + 1.$
15. $3a^2 + 4ab + b^2.$
16. $2x^3 - 5x^2 - 2x + 9$, Rem. $9x - 24.$
17. $3x^4 - 5x^3 + 2x + 3$, Rem. $1.$
18. $a^2 - ab + b^2.$
19. $a^4 + a^2b^2 + b^4.$
20. $a^{10} - a^5b^5 + b^{10},\ a^{12} - a^9b^3 + a^6b^6 - a^3b^9 + b^{12}.$
21. $a^5 - b^5,\ a^3 - b^3.$
22. $x^4 + 2x^3 + 3x^2 + 2x + 1,\ x^2 + x + 1.$
23. $x^4 + x^2 + 1,\ x^3 + 2x^2 + 2x + 1.$
24. $a^2 + 2ab + b^2.$
25. $x^3 - 2x^2y + 2xy^2 - y^3.$
26. $-8a^4 + 8a^2b^2 + 3ab^3 + b^4.$
27. $x^3 + px.$
28. $x^2 + mx - n.$
29. $a^2x^2 + 2abxy + b^2y^2.$
30. $3a^2x - 4ax^2 + 16x^3.$
31. $x^2 - 3x + 2.$
32. $x^5 - 2x^4 + 8x^2 - 16x.$
33. $x^3 + 2x^2 + 3x - 1.$
34. $a^2 + 2ab + b^2.$
35. $a + b + 3.$
36. $a^2 + b^2.$
37. $a^2 - ab + b^2 + a + b + 1.$
38. $a^2 - 2a + 1.$
39. $x^3 + 4x^2 + 8x + 6$, Rem. $13x - 10.$
40. $x^3 - 5x^2 + 7x.$

Exercise XV. Page 63

1. $(a+m+p)x + (b+n+q)y.$
2. $(4a+c)x + (7-3b)y.$
3. $(m-1)x - (x-1)y.$
4. $(a^2+ab+b^2)x + (a^2-2ab-b^2)y.$

5. $(b+1)x+(a+4)y$. 6. $nx-my$. 7. $-nx$.
8. $2(a-c)x-2(a-b)y$. 9. $(a-p)x^2+(b+q)xy-(c+r)y^2$.
10. $(c-b)x^2+(a+b)xy-(c-b)y^2$.
11. $(a-b-1)x+(b-1)y-(a-1)z$.
12. $x^2+(a+b)x+ab$. 13. $x^2-(a+b)x+ab$.
14. $x^2+(a-b)x-ab$. 15. $x^2-(a-b)x-ab$.
16. $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$.
17. $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$.
18. $x^3+(a-b+c)x^2-(ab+bc-ac)x-abc$.
19. $x^3-(a-b+c)x^2-(ab-ac+bc)x+abc$.
20. $3x^2-(ab+bc+ca)$. 21. $a+b-c$.
22. $a^2+b^2+c^2-ab-ac-bc$. 23. $4a^2+b^2+c^2+2ab-2ac+bc$.
24. $x+2y-z$. 25. $x-b$. 26. $x-a$.
27. $x^2-mx+mn$. 28. $x^2+(b+c)x+bc$, $x^2+(a+c)x+ac$.
29. $x^2-(a+c)x+ac$, $x^2-(a+b)x+ab$. 30. $x+b$.
31. $a(b+c)-2bc$. 32. $a^2+2ab+b^2-c^2$, $-a^2+b^2+2bc+c^2$.

Exercise XVI. Page 69

1. $x=12$. 2. $x=9$. 3. $x=27$. 4. $x=47$.
5. $x=2$. 6. $x=2$. 7. $x=-16$. 8. $x=-1$.
9. $x=-3$. 10. $x=2$. 11. $x=0$. 12. $x=6$.
13. $x=-8$. 14. $x=-2\frac{1}{4}$. 15. $x=31$. 16. $x=98$.
17. $x=3$. 18. $x=\frac{3}{10}$. 19. $x=15$. 20. $x=3$.
21. $x=4\frac{1}{3}$. 22. $x=2$. 23. $x=8a$. 24. $x=a+b$.
25. $x=0$. 26. $x=a+b$. 27. $x=\frac{a}{2}$. 28. $x=\frac{b}{2}$.
29. $x=1$. 30. $x=b-a$. 31. $x=a+b$. 32. $x=b-a$.
33. $x=c-a-b$. 34. $x=a+b$. 35. $x=c$. 36. $x=a+b+c$.

Exercise XVII. Page 72

1. 12. 2. 46, 29. 3. 10. 4. 71 cts., 54 cts.
5. 57 cts. 6. 11 ft. 7. 19 ft. 8. $13.
9. 30 yrs. 10. 40. 11. 25, 17, 14. 12. 39, 36.

13. 452, 423.
14. 24, 29, 36.
15. \$1.88, \$1.93, \$3.83.
16. \$100.
17. 55, 35.
18. 10
19. 28, 5.
20. 21.
21. \$560, \$460.
22. 19, 24.
23. 54 cts., 40 cts.
24. 20 cts.
25. 40 cts.
26. 40 cts.
27. 85, 51, 33.
28. \$3.20.
29. 15.
30. \$3.80.
31. 60.
32. \$67.
33. 23.
34. \$5.00.
35. $41\frac{2}{3}$.
36. 60 gals.
37. 400 sq. in.
38. 324 sq. in.
39. 12800 sq. yds.

Exercise XVIII. Page 78

1. $x=6$.
2. $x=7\frac{2}{3}$.
3. $x=12$.
4. $x=3$.
5. $x=1$.
6. $x=3\frac{1}{4}$.
7. $x=\frac{1}{3}$.
8. $x=3\frac{5}{6}$.
9. $x=2$.
10. $x=9$.
11. $x=1$.
12. $x=\frac{3}{4}$.
13. $x=\frac{9}{16}$.
14. $x=-5$.
15. $x=0$.
16. $x=31$.
17. $x=2$.
18. $x=2\frac{1}{4}$.
19. $x=\frac{8}{11}$.
20. $x=4$.
21. $x=2$.

Exercise XIX. Page 81

1. 17.
2. 21.
3. 24.
4. 9, 8.
5. 33.
6. $1\frac{1}{4}$ mi.
7. 40.
8. 39.
9. 36 ft.
10. 75 cts.
11. 40.
12. 149.
13. 16, $10\frac{1}{2}$.
14. 18.
15. \$350, \$650.
16. \$480, \$300.
17. 21.
18. 54.
19. 19, 24.
20. 10 yrs.
21. 108 sq. ft.
22. 15.
23. \$600.
24. \$1200.
25. \$160.
26. \$48
27. \$12.
28. \$800.

Exercise XX. Page 88

1. $x=12$, $y=8$.
2. $x=22$, $y=3$.
3. $x=11$, $y=13$.
4. $x=13$, $y=23$.
5. $x=3$, $y=-5$.
6. $x=2$, $y=7$.
7. $x=5$, $y=4$.
8. $x=8$, $y=3$.
9. $x=4$, $y=-2$.
10. $x=-4$, $y=3$.
11. $x=6$, $y=10$.
12. $x=15$, $y=10$.

13. $x=3$, $y=5$,
14. $x=11$, $y=2\frac{1}{2}$.
15. $x=5$, $y=-1$.
16. $x=22$, $y=24$.
17. $x=-3$, $y=2$.
18. $x=\frac{1}{3}$, $y=\frac{1}{4}$.
19. $x=2$, $y=\frac{1}{2}$.
20. $x=16$, $y=-3$.
21. $x=3$, $y=1$.
22. $x=2$. $y=-5$.

Exercise XXI. Page 91

1. 19, 28
2. 12, 8.
3. 40 cts., 21 cts.
4. \$2.25, \$1.20.
5. 30 cts., 40 cts.
6. 10 lbs.
7. \$22, \$26.
8. \$12, \$15.
9. \$1.20, \$0.60.
10. 180, 200.
11. \$3.50, \$2.25.
12. $12+20=32$.
13. 125.
14. 36.
15. 36.
16. 26.
17. The sum of the digits ; the difference of the digits.
18. When the large digit is in the tens' place, the number is 7 times the sum of its digits.
19. $\frac{3}{8}$.
20. $\frac{5}{16}$.
21. 28.
22. 24.
23. 20 ft., 12 ft.
24. $2\frac{3}{4}$ miles.
25. 27 miles.

EXAMINATION PAPERS

I. Page 94

1. 16.
2. $x+y+x+y$, $2x+2y$, $2(x+y)$; xy, $144xy$.
3. $-4a+7b-7c$.
4. a^3+a^2b.
5. a^2-ab+b^2.
6 $3\frac{1}{2}$.
7. 30 lbs.

II. Page 95

1. $-2\frac{1}{3}$.
2. 12, 64, 81.
3. $-2\frac{1}{2}ab+\frac{1}{3}b^2$.
4. $2x^2-15x-8$.
5. $a-10b$.
6. $x=3$.
7. $7\frac{1}{2}$ weeks, \$75, \$82.50, \$90.

III. Page 95

1. $\frac{1}{6}$,
2. 9 sq. ft.
3. Twice the larger number; twice the smaller number.
4. 0, $2\frac{1}{3}$.
5. $x-a$.
6. 20.
7. 30, 50.

Answers and Results

IV. Page 96

1. $3\frac{1}{8}$.
2. $8x-4,\ 4x^2-4x-15,\ 4x^2+4x+1$.
3. $2a-5b+3b^2$.
4. $2a^3+2b^3+2c^3-6abc$.
5. a^2-ab+b^2.
6. $x=15$.
7. 10 years.

V. Page 97

1. 120.
2. a^2+3b^2.
3. -1.
4. 0.
5. $\frac{2}{7}$
6. $x=5,\ y=2$.
7. 18, 6.

VI. Page 97

1. $-\frac{1}{3}$.
2. $a^4+a^2b^2$.
3. 1.
4. $ab+bc+ac$.
5. $x=2$.
6. $x=4,\ y=9$.
7. 19, 47.

VII. Page 98

1. $z=5$.
2. 2.
3. $2(a^2b+a^2c+ac^2+bc^2+abc-b^3)$.
4. $1-b+ab$.
5. $x=a^2+ab+b^2$.
6. 12 ft., 10 ft.
7. 24, 19.

VIII. Page 99

1. $3a^2+ab+4b^2$.
2. x^2-a^2.
3. $ax+by-1$.
4. $9(ab+bc-2ac-2c^2)$.
5. $x=8$.
6. 37.
7. \$200, \$30.